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ORIGINAL PAPER

The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability

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Abstract A brief historical overview is given which discusses the development of classical stability concepts, starting in the seventeenth century and finally leading to the concept of Lyapunov stability at the beginning of the twentieth century. The aim of the paper is to find out how various scientists thought about stability and to which extent their work is related to the stability concepts bearing their names, i.e. Lagrange, Poisson and Lyapunov stability. To this end, excerpts of original texts are discussed in detail. Furthermore, the relationship between the various works is addressed.

Keywords History of mechanics · Stability theory · Torricelli's axiom · Lyapunov · Lagrange–Dirichlet stability theorem

1 Introduction

Is the solar system stable? Under which load will a beam buckle? Is the figure of equilibrium of a steady rotating fluid stable? These fundamental questions were some of the major problems that motivated scientists such as Euler, Lagrange, Poincaré and Lyapunov to think about the concept of stability of motion. The origin of stability theory must clearly be sought in mechanics. The interest in the stability of motion is today greater than ever and is no longer confined to mechanics. Stability issues play a role in economical models, numerical algorithms, quantum mechanics, nuclear physics, and control theory as fruitfully applied in for example the fields of mechanical and electrical engineering.

Various stability concepts exist in modern literature, e.g. Lagrange, Poisson and Lyapunov stability. Apparently, these stability concepts bear the names of illustrious scientists from the past. The reason why these names are connected with these notions of stability is often far from trivial. One might be tempted to make the possible erroneous assumption that a stability notion has been developed by the scientist after which it is named. A historical survey on how these stability concepts came into being therefore seems appropriate.

The existing literature on the history of mechanics has adequately discussed the life and work of leading scientists and has studied the history of various scientific problems and developments which have shaped the scientific progress in mechanics, such as the three-body problem, the development of the concept of force or the invention of calculus. However, the particular history of stability concepts has barely been addressed. One of the few publications on this topic is the historical overview of Loriá and Panteley [32], which focuses

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on the post-Lyapunov era. The concept of stability is used in many different scientific fields and the word ‘stability’ is not only used in the context of mechanics. A complete historical survey of stability concepts would therefore have to involve an in-depth study of a great variety of contexts such as philosophy, theology, astronomy, numerical analysis, structural mechanics, fluid dynamics and control theory, as well as the circulation of ideas and techniques between these domains. The scope of the present paper is much more limited.

The aim of this paper is to give a short historical account of the development of the main classical stability concepts which are used in the theory of dynamical systems and modern control theory. A limited number of contexts have been chosen which have directly influenced modern stability theory. Furthermore, the scope has been limited with respect to time. One may go back in history as far as Aristotle and Archimedes to study the origin of stability ideas, but the Age of Reason (the seventeenth century) has been chosen as starting point of the historical survey. The formulation of Lyapunov’s stability concept in the beginning of the twentieth century marks the beginning of modern stability theory and has therefore been chosen as endpoint. In particular, the aim of this paper is to find out how various scientists thought about stability and to which extent their work is related to the stability concepts bearing their names. To this end, excerpts of original texts are presented in their original form and language, translations are given, and they are subsequently discussed in detail. Furthermore, the relationship between the various works is addressed.

2 Torricelli’s axiom

The Aristotelian idea that weight strives to a natural position in the centre of the world (i.e. the universe) led in the fourteenth century to the concept of *centrum gravitas*, which encompasses the idea that every body possesses a point in which all heaviness can be thought to be concentrated and which has the tendency to coincide with the centre of the world [9]. Galileo Galilei (1564–1642) altered this Aristotelian idea by replacing the centre of the world by that of the centre of the earth. Evangelista Torricelli (1608–1647), a disciple of Galilei, put these ideas into a more mathematical formulation. In his work *De Motu Gravium*, Torricelli postulated his axiom (reproduction from [46], p. 99):

> **Duo grauia ſimul coniuncta ex ſe moueri non poſſe, niſi centrum commune grauitatis ipſorum deſcendat.**

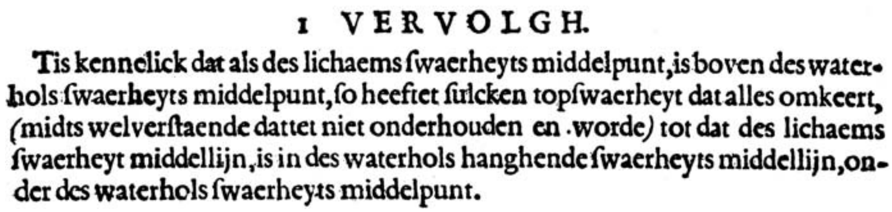
Translation:

> *Two weights which are linked together cannot start moving by themselves if their common centre of gravity does not descend.*

In this axiom, Torricelli considers a system of two interconnected weights being in equilibrium under the influence of gravity. If the equilibrium is in modern terms unstable, then a small initial disturbance leads to a considerable movement of the system, i.e. the weights of the system can ‘start moving by themselves’. If, however, the equilibrium is stable, then a small disturbance does not lead to a considerable movement. Torricelli simply uses the words ‘start moving by themselves’ without speaking of an initial disturbance from an equilibrium. A modern interpretation of Torricelli’s axiom is that a mechanical system of interconnected rigid bodies subjected to gravity with sufficient kinematical constraints and remaining degrees of freedom attains a static equilibrium in such a configuration that the centre of gravity of the system takes the lowest possible position (at least locally). This is necessarily a stable equilibrium, because one has to lift the centre of gravity and increase the total potential energy of the system in order to change the configuration of the system, being the modern static stability criterion. Hence, although Torricelli did not use the word ‘stability’, his axiom certainly preludes a stability concept based on the (gravitational) potential energy.

3 Stability of floating bodies

Inspired by the work of Archimedes of Syracuse, the Flemish and Dutch scientists Simon Stevin (1548–1620) and Christiaan Huygens (1629–1695) studied the equilibrium of floating bodies. In his work *Byvough der weeghconst* [42], Stevin wrote (following Archimedes) that a floating body takes such a position that its centre of gravity is on the vertical centre line


1 VERVOLGH.

Tis kennelick dat als des lichaems ſwaerheyts middelpunt,is boven des water-hols ſwaerheyts middelpunt,ſo heeftet ſulcken topſwaerheyt dat alles omkeert, (midts welverſtaende dattet niet onderhouden en .worde) tot dat des lichaems ſwaerheyt middellijn,is in des waterhols hanghende ſwaerheyts middellijn,onder des waterhols ſwaerheyts middelpunt.

Fig. 1 Corollary of Stevin, *Byvough der weeghconst* (reproduction from [42], p. 202)

of gravity of the displaced fluid. Furthermore, he gives the following corollary, see Fig. 1:

Translation (from [7]):

> *It is obvious that if the body's centre of gravity is above the centre of gravity of the body of the displaced water, it has such top-heaviness that everything turns over* (*provided, however, it be not supported*) *until the body's centre line of gravity is in the vertical centre line of gravity of the body of displaced water, below the centre of gravity of the body of displaced water.*

Apparently, Stevin thought that a floating body is stable when the centre of gravity is below the centre of buoyancy ('the centre of gravity of the body of the displaced water'), which is not generally true. Compared with Stevin, the work of Huygens [17] has a more mathematical style, but still relies upon geometrical methods as was common in the seventeenth century. In many theorems, such as the following one, Huygens addresses stability problems of floating bodies (cited from [17], Theorema 6, p. 103):

> *Si corpus solidum liquido supernatans ultrò inclinetur et alium situm acquirat; altitudo centri gravitatis totius corporis supra centrum gravitatis partis mersae, minor erit positione corporis posteriori quam priori.*

Translation:

> *If a solid body, floating on a liquid, inclines and acquires another position, then the height of the centre of gravity of the total body over the centre of gravity of the submerged part will be smaller in the latter position than in the former position.*

The theorems of Huygens have a strong similarity with the work of Stevin. However, the novelty in the work of Huygens is that he explicitly compares two different positions of the system. In the 18th century, the study of the roll-stability of ships was carried on by Daniel Bernoulli (1700–1782), Leonhard Euler (1707–1783) and Pierre Bouguer (1698–1758). Daniel Bernoulli distinguishes between stable (which he calls 'firm') and unstable equilibria of floating bodies and writes (cited from [3], Sect. 3 on p. 148):

> [. . .] *quo ambo aequilibrii situs ab invicem distinguuntur; minima quidem vis quaevis corpora etiamsi in aequilibrio firmo posita aliquantillum nutare facit, sublata autem vi corpus rursus ad situm naturalem tendit, nisi nutatio certos quosdam terminos transgressa fuerit.*

Translation (by F. Cerulus, private communication, to be published in The Bernoulli Edition, vol. 6 of the 'Werke von Daniel Bernoulli'):

> [. . .] *by this, both positions of equilibrium are distinguished one from the other; indeed, a minimal arbitrary force makes a body—although put in firm equilibrium—nod a little, but when the force has been undergone* [*i.e. ceases to act*], *the body tends again to its natural position, unless the nodding would have exceeded certain bounds.*

Daniel Bernoulli speaks explicitly of the stability of an *equilibrium* and considers the couple of restoring forces when the equilibrium of the floating body is perturbed by a small amount. Similarly to Huygens, Bernoulli implicitly considers two different positions of the system but their distance is small. Euler refines the work of Daniel Bernoulli in his two-volume treatise *Scientia Navalis* [11] and distinguishes between equilibria which are stable, unstable and indifferent (cited from [11], vol. 1, Chap. 3, Proposition 19, p. 86):

> *Stabilitas, qua corpus aquae innatans in situ aequilibrii persuerat, aestimanda est ex momento potentiae restituentis, si corpus dato angulo infinite paruo ex situ aequilibrii fuerit declinatum.*

Translation:

> *The stability of a floating body in equilibrium is determined by the restoring moment arising*

> *when the body has been displaced from equilibrium by an infinitesimally small angle.*

Euler uses the word 'stability' and associates stability with the response on an *infinitely small* disturbance from the equilibrium position. The idea of infinitely small disturbances will later play a role in the work of Lagrange. Bouguer [6], working independently from Euler, introduced the term 'metacentric height', which became the modern expression to determine the roll-stability of ships [38].

4 Elastic stability

The theory of elastic stability in statics began with the work of Euler on the critical buckling load of columns. Daniel Bernoulli suggested in a letter to Euler (dated October 20th 1742, published as 'Lettre XXVI' in [14]) that the differential equation of the *elastica* could be found by minimising the integral of the square of the curvature along the rod, being proportional to what we now call the elastic strain energy. Euler acted on this suggestion in his 'Additamentum' *De curvas elasticis* of his work *Methodus inveniendi lineas curvas maximi minive proprietate gaudentes* [10] (see also the German translation in [30]) on the calculus of variations. After deriving the differential equation of the elastica by minimising the elastic strain energy using the calculus of variations, Euler proceeds to derive the same equation by known mechanical principles, thereby establishing the validity of the variational procedure (see the historical review in [12]). Based on the differential equation for the elastica, Euler found a certain length which a column must attain to be bent by its own weight or an applied weight, and concluded that for shorter lengths it will simply be compressed, while for greater lengths it will be bent, i.e. buckle. Although Euler initiated the analysis of the elastic stability of the static equilibrium, he tacitly left the notion of stability undefined. In this context, we have to remark that stability is in essence a concept of dynamical systems as time plays an essential role.

5 The Lagrange–Dirichlet stability theorem

The development of a stability concept in dynamics was continued by J.L. Lagrange (1736–1813), who formalised the axiom of Torricelli for conservative dynamical systems employing the concept of potential energy. In his monumental work *Méchanique Analytique*, Lagrange wrote (cited from [21], Part 1, Sect. 3, No. 16, p. 38):

> *On vient de voir que la fonction Φ* [*the potential energy*] *est un minimum ou un maximum, lorsque la position du systême est celle de l'équilibre ; nous allons maintenant démontrer que si cette fonction est un minimum, l'équilibre aura de stabilité ; ensorte que le systême étant d'abord supposé dans l'état de l'équilibre, & venant ensuite à être tant soit peu déplacé de cet état, il tendra de lui-même à s'y remettre, en faisant des oscillations infiniment petites.*

Translation:

> *We have shown that the* [*potential energy*] *function Φ is in a minimum or maximum, when the configuration of the system is one of equilibrium; we are now going to demonstrate that if this function is in a minimum then the equilibrium will be stable, such that the system, being assumed in equilibrium and displaced by a small amount, will tend to return to it by itself while making infinitely small oscillations.*

In other words, Lagrange posed the theorem that, if the system is conservative, a state corresponding to zero kinetic energy and minimum potential energy is a stable equilibrium point. Moreover, Lagrange gave a definition of stability of an equilibrium. Clearly, Lagrange meant that an equilibrium is stable when neighbouring solutions remain close to the equilibrium, which agrees with our modern concept of stability in the sense of Lyapunov. Lagrange speaks of infinitely small oscillations around an equilibrium because a stable equilibrium in a conservative system is necessarily a centre. Using a Taylor series approximation of the potential energy up to second-order terms, Lagrange proved that the equilibrium is indeed stable when the first-order terms vanish and the second-order terms are positive, corresponding to a minimum of the potential energy. J.P.G. Lejeune Dirichlet (1805–1859) added a note [27] to the theorem of Lagrange, arguing that a minimum of the potential energy might also be caused by fourth or higher-order terms in the Taylor series, but a minimum of the potential energy is sufficient to prove stability. The theorem is in literature therefore referred to as the Lagrange–Dirichlet stability theorem and plays an important role in elasto-statics. An equilibrium in elasto-statics is called stable when it corresponds to a minimum of the potential energy. Hence, the Lagrange–Dirichlet stability theorem is used in elasto-statics as the *definition* of stability (instead of a condition for stability). The reason for this is that the notions of time, velocity and trajectory are non-existent in statics and a definition of stability based on those notions cannot be given.

The stability concept employed by Lagrange was adopted in the scientific world. P.S. Laplace (1749–1827), for instance, used a very similar notion of stability (cited from [25], Livre 3, p. 164):

> *Il existe deux états très-distincts d'équilibre ; dans l'un, si l'on trouble un peu l'équilibre, tous les corps du systême, ne font que de petites oscillations autour de leur position primitive, et alors l'équilibre est ferme ou stable. Cette stabilité est absolue, si elle a lieu quelles que soient les oscillations du système ; elle n'est que relative, si elle n'a lieu que par rapport aux oscillations d'une certaine espèce. Dans l'autre état d'équilibre, les corps s'éloignent de plus en plus de leur position primitive, lorsqu'on vient à les en écarter.*

Translation:

> *There exist two very distinct states of equilibrium; in one, if we perturb a little the equilibrium, all bodies of the system only make small oscillations around their primitive position and the equilibrium is therefore firm or stable. This stability is absolute if it is present whatever oscillations of the system may be; it is only relative, if it is present with respect to oscillations of a certain type. In the other state of equilibrium, the bodies move away from their primitive position when separated from it.*

6 The stability of a rotating fluid

The problem of the figure of the earth led in the late 17th century to the question of what the possible figures of equilibrium are of a rotating fluid. A homogeneous mass of fluid was considered, which is rotating around an axis through its centre of gravity, under the influence of no forces beyond centrifugal force and the mutual gravitational attraction of its molecules when it is assumed that the whole fluid rotates as if it were solid (see the historic survey [26]). It was found that the figure is either a 'MacLaurin ellipsoid', which is an ellipsoid with rotational symmetry, or a 'Jacobi ellipsoid' for which all axes are different. J. Liouville (1809–1882) studied the stability of figures of equilibrium by using a modified version of the Lagrange–Dirichlet stability theorem [31]. Liouville stated that, in the absence of dissipation, the figure of equilibrium is stable if it corresponds to a maximum of the *force vive*, being twice the kinetic energy. A maximum of the kinetic energy corresponds to a minimum of the potential energy if the total energy is conserved. The work of Liouville later inspired Poincaré and Lyapunov.

Hydrodynamic instability became a central question in the nineteenth century. The history of stability concepts in general fluid mechanics is beyond the scope of this paper, but has been studied in detail in [8].

7 The stability of the world system

Celestial mechanics has greatly influenced the terminology of modern stability theory (for a good overview of this topic see [45]). P.S. Laplace (1749–1827) studied the celestial three-body problem using a perturbation analysis neglecting terms in the mass of second-order and higher and assuming small values of eccentricity [24]. He concluded that, under these assumptions, the variation of the semi-major axis of the orbits is periodic with a constant amplitude, i.e. of the form $A\sin(\alpha t + \beta)$. Laplace makes the following conclusion (cited from [23], p. 248–249):

> *Ainsi le système du monde ne fait qu'osciller autour d'un état moyen dont il ne s'écarte jamais que d'une très petite quantité. Il jouit, en vertu de sa construction et de la loi de la pesanteur, d'une stabilité qui ne peut être détruite que par des causes étrangères* [...]

Translation:

> *Thus the world system only makes small oscillations around an average state of which it never deviates more than a small amount. It enjoys, by virtue of its construction and the law of gravitation, a stability which can only be destructed by external causes* [...]

Hence, Laplace speaks of the *stabilité du système du monde*, the stability of the world system. It should be emphasised that Laplace's definition of stability for the solar system concerns a single motion with given initial conditions, whereas most other notions of stability, such as of Lagrange presented in Sect. 5, imply perturbations of the initial conditions.

Though Laplace is traditionally given credit for establishing the proof of the stability of the solar system, it is only after Lagrange's work that Laplace made his first major contribution to the theory of the stability of the solar system. Laplace's original contribution [22] is limited and non-rigorous. It was Lagrange [20] who invented a general method which extended the result of Laplace for arbitrary eccentricities, still neglecting terms in the mass of second-order and higher. S.D. Poisson (1781–1840) extended the perturbation analysis to second as well as to third-order terms in the mass. Poisson showed that the variation of the semi-major axis only contains terms of the form $A\sin(\alpha t+\beta)$ if terms in the mass of second-order are taken into account [40]. Based on this analysis, the motion of the earth and moon with respect to the sun therefore remains bounded and Poisson writes (cited from [40], p. 5):

> [...] *nous avons fait voir que la stabilité du système planétaire n'est pas altérée, lorsqu'on a égard aux carrés des masses et à toutes les puissances des excentricités et des inclinaisons.*

Translation:

> [...] *we have shown that the stability of the planetary system is not changed if one takes into account the square of the masses and all powers of eccentricities and inclinations.*

Similarly to Laplace, Poisson associates the boundedness of the variation of the semi-major axis with the concept of stability.

The extension of Poisson to third-order terms in the perturbation analysis reveals in addition secular terms of the form $At\sin(\alpha t+\beta)$ in the variation of the semi-major axis. The amplitude of these oscillatory terms grows unboundedly with time. An analysis strictly based on third-order terms, and neglecting higher-order terms, therefore implies that the motion of the planets does not remain bounded but the planets come arbitrarily close to their original position infinitely many times.

Also C.G.J. Jacobi (1804–1851) contemplates whether the results of Laplace, Lagrange and Poisson are the proof of the *Stabilität des Weltsystems*, i.e. the *stability* of the world-system (cited from [18], pp. 29–30):

> *In diesen und ähnlichen Betrachtungen liegt der Kern der berühmten Untersuchungen von* Laplace, Lagrange *und* Poisson *über die Stabilität des Weltsystems. Es existirt nämlich der Satz: Nimmt man die Elemente einer Planetenbahn veränderlich an und entwickelt die grosse Axe nach der Zeit, so tritt diese nur als Argument periodischer Functionen ein, es kommen keine der Zeit proportionale Terme vor.*

Translation:

> *In these and similar considerations lies the essence of the famous studies of* Laplace, Lagrange *and* Poisson *on the stability of the world system. Namely, the statement exists: If one assumes the elements of a planetary orbit to be variable and if one expands the semi-major axis in time, then time only appears as argument of periodic functions, i.e. no terms proportional to time appear.*

We conclude that Jacobi, like Laplace and Poisson, associated the word stability with periodic functions which do not grow unboundedly. Jacobi continues with a discussion of secular terms in the third-order perturbation analysis of Poisson and finally remarks that the truncation of the order in the perturbation analysis does not allow for a conclusion about the boundedness of the motion.

In 1887, King Oscar II of Sweden sponsored a mathematical competition with a prize for a resolution of the question of how stable the solar system is, a variation of the three-body problem. J.H. Poincaré (1854–1912) won the prize, being the starting point of his work on the stability of the solar system which accumulated in his work *Les Méthodes Nouvelles de la Mécanique Céleste* [39] (for a historical survey on Poincaré, see [1, 15]). Poincaré states [39] that the term 'stability' is used in different ways and he continues to discuss the differences between the results of Lagrange and Poisson on the variation of the semi-major axis. According to Poincaré, Lagrange found that the variation of the semi-major axis is governed by terms of the form $A\sin(\alpha t+\beta)$, whereas Poisson found that there are in addition terms of the form

$At \sin(\alpha t + \beta)$. He comes to the conclusion (cited from [39], vol. III, Chap. XXVI, pp. 140–141):

> *Le mot de* stabilité *n'a donc pas le même sens pour Lagrange et pour Poisson.*

Translation:

> *The word* stability *does therefore not have the same meaning for Lagrange and for Poisson.*

With respect to the solar system, Poincaré speaks of the stability in the sense of Lagrange, with which he means bounded behaviour of the planetary orbits, and stability in the sense of Poisson, for which the planets come arbitrarily close to their original position infinitely many times. Hence, Poincaré gives the false impression that Lagrange associates his work on the semi-major axis with the concept of stability. Similarly, reading Poincaré, one may falsely conclude that Poisson's notion of stability is related to non-periodic terms which grow unboundedly. These two errors have persisted in modern times and have led to misnomers. Nowadays, the terms 'Lagrange-stability' and 'Poisson-stability', inherited from Poincaré, are used to denote boundedness and recurrence of solutions of arbitrary dynamical systems. The modern concept of Lagrange stability is therefore very different from the concept of stability as expressed by Lagrange himself in *Méchanique Analytique* [21]. The work of Poincaré on periodic orbits led to the modern concept of Poincaré stability, which is also called orbital stability.

8 The stability of regulators

In the nineteenth century, the development of regulators for steam engines and water turbines led to the stability analysis of machines and regulators. The history of automatic control has been studied in detail in the field of Systems and Control Theory (see for instance [2, 4, 5, 19]). In this paper we will therefore only briefly touch this subject.

J.C. Maxwell (1831–1879) analysed the stability of Watt's flyball governor [37] (see the historical survey [16]). His technique was to linearise the differential equations of motion to find the characteristic equation of the system. He studied the effect of the system parameters on stability and showed that the system is stable if the roots of the characteristic equation have negative real parts. In 1877, E.J. Routh (1831–1907) provided an algorithm for determining when a characteristic equation has stable roots [41]. Around the same time, the Russian И.А. Вышнеградский (1831–1895), transliterated as I.A. Vyshnegradsky, analysed the stability of regulators using differential equations independently of Maxwell and studied the stability of the Watt governor in more detail [47]. A.B. Stodola (1859–1942) studied in 1893 the regulation of a water turbine using the techniques of Vyshnegradsky [43, 44]. Stodola modelled the actuator dynamics and included the delay of the actuating mechanism in his analysis and was the first to mention the notion of the system time constant. Unaware of the work of Maxwell and Routh, Stodola posed the problem of determining the stability of the characteristic equation to A. Hurwitz (1859–1919), who solved it independently [4].

9 Lyapunov stability

Exact mathematical definitions of stability for a dynamical system, as well as general stability theorems for nonlinear systems, were first formulated by Russian scientists at the end of the nineteenth century. The Russian scientist Н.Е. Жуковский (1847–1921), transliterated as N.E. Zhukovskii, introduced in 1882 a strong orbital stability concept which is based on a reparametrisation of the time variable [48]. The work of Zhukovskii has been almost forgotten and has received renewed attention only recently [28]. The stability concept of Zhukovskii agrees with Poincaré stability when only equilibria and periodic solutions are considered, which might explain why Zhukovskii's ideas fell into oblivion [28]. Moreover, the great success of the work of Lyapunov might have overshadowed the contribution of Zhukovskii. In 1892, ten years after the work of Zhukovskii, the Russian scientist A.M. Ляпунов (1857–1918), usually transliterated as A.M. Lyapunov, defended his PhD thesis *A general task about the stability of motion* [35]. The work of Lyapunov became famous in Russia and later also in the West. The PhD thesis of Lyapunov [35] was reprinted in Russian in 1950 [36]. It was translated into French in 1907 [29] and this translation was reproduced in [33]. An English translation and biography has been published in 1992 in the centenary issue [13]

Fig. 2 Definition of stability according to Lyapunov (reproduction from [36], pp. 19–20)

Пусть $L_1, L_2, \ldots, L_n$ суть произвольно задаваемые положительные числа. Если при всяких L_s, как бы они малы ни были, могут быть выбираемы положительные числа $E_1, E_2, \ldots, E_k, E'_1, E'_2, \ldots, E'_k$ так, чтобы при всяких вещественных ε_j, ε'_j, удовлетворяющих условиям

$$|\varepsilon_j| \leqslant E_j, \quad |\varepsilon'_j| \leqslant E'_j \text{ [1])} \qquad (j = 1, 2, \ldots, k),$$

и при всяком t, превосходящем t_0, выполнялись неравенства

$$|Q_1 - F_1| < L_1, \quad |Q_2 - F_2| < L_2, \ldots, |Q_n - F_n| < L_n,$$

то невозмущенное движение по отношению к величинам $Q_1, Q_2, \ldots, Q_n$ устойчиво; в противном случае— неустойчиво [4].

in memory of Lyapunov and in [34]. Lyapunov himself reviewed and corrected the French version and introduced some additional material [34]. The English version [13, 34] is a translation of the French version of 1907. In the following we will study the Russian version of 1950 [36].

The notation of Lyapunov differs a little from the modern notation. Lyapunov introduces n quantities F_i which are functions of the k trajectories $f_j(t)$ of the positions q_j starting from the unperturbed initial condition q_{j0}. The quantities Q_i denote these functions for the perturbed trajectories due to perturbations ε_j on the initial position and ε'_j on the initial velocity. The definition of stability, according to Lyapunov reads in the original Russian version [36] as, see Fig. 2:

Translation:

Let $L_1, L_2, \ldots, L_n$ be given arbitrary positive numbers. If for all L_s, no matter how small they are, one can choose positive numbers $E_1, E_2, \ldots, E_k$, $E'_1, E'_2, \ldots, E'_k$, such that for all real ε_j, ε'_j satisfying the conditions

$$|\varepsilon_j| \leqslant E_j, \quad |\varepsilon'_j| \leqslant E'_j \quad (j = 1, 2, \ldots, k)$$

and all t, greater than t_0, the following inequalities are satisfied

$$|Q_1 - F_1| < L_1, \quad |Q_2 - F_2| < L_2, \ldots, |Q_n - F_n| < L_n,$$

then the non-perturbed motion is stable with respect to the quantities $Q_1, Q_2, \ldots, Q_n$; otherwise—unstable.

In modern terminology, an equilibrium is defined to be Lyapunov-stable if for each ε-neighbourhood one can find a δ-neighbourhood of initial conditions, such that their solutions remain within the ε-neighbourhood. Note that the quantities L_i construct the ε-neighbourhood in the original definition of Lyapunov, while the quantities E_j set up the δ-neighbourhood. The original definition of Lyapunov is for a mechanical system with k degrees of freedom and with respect to n given functions Q_i of the k positions q_j. The modern definition of Lyapunov stability is for arbitrary dynamical systems and is no longer restricted to mechanical systems. Furthermore, the modern definition of Lyapunov stability is not with respect to some functions on the state of the system, but to the state itself. Loosely speaking, Lyapunov stability means that neighbouring solutions remain close to the equilibrium, which is essentially the same as what Lagrange understood under the term stability (see the citation of Lagrange in Sect. 5). Lyapunov proved stability using two distinct methods. In the first method, known as Lyapunov's first method or Lyapunov's indirect method, the stability of an equilibrium is studied through linearisation. The second method, also called the direct method of Lyapunov, is far more general. The fundamental idea behind the direct method of Lyapunov is the stability theorem of Lagrange–Dirichlet, which is based on the mechanical energy. The direct method of Lyapunov is able to prove the stability of equilibria of nonlinear differential equations using a generalised notion of energy functions. Unfortunately, though his work was applied and continued in Russia, the time was not ripe in the West for his elegant theory, and it remained un-known there until its French translation in 1907. Its importance was finally recognised in the 1960s with the emergence of control theory.

10 Closure

Various excerpts of original texts on stability issues have been studied in the previous sections and have been put in their historical context. The above study is very limited and fragmentary. Still, we can draw a few tentative conclusions.

The technical term 'stability' in the context of mechanics already appeared in 1749 in the work of Euler. The question of roll-stability of floating bodies has been a strong motivation for the theoretical research on stability in the seventeenth and eighteenth century.

Celestial mechanics has influenced foremost the terminology in stability theory. The modern concepts of Lagrange and Poisson stability are due to Poincaré and have little to do with how Lagrange and Poisson actually thought about stability. Hence, from a historical point of view, one might regard 'Poisson stability' and 'Lagrange stability' as misnomers.

Interestingly, the essence of the Lyapunov stability concept, which is usually thought to be developed at the end of the nineteenth century, can already be found in the work of Lagrange. The idea of solutions which remain in the neighbourhood persisted through the ages and can also be found in for instance the work of Laplace and Poisson. The exact mathematical definition of stability in terms of an ε–δ technique is most probably due to the nineteenth century Russian school, but the origin of the general idea behind this stability concept must be sought in the eighteenth century in western Europe.

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