The Euler-Maupertuis principle of least action as variational inequality

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Starting from Hamilton's Principle, the current paper discusses how we can derive the Euler-Maupertuis Principle of Least Action in the context of non-smooth dynamics. This variational principle allows us to *directly* obtain the space curve $y(x)$ of a point-mass in a potential field $V(x, y)$ without referring to the temporal dynamics. This paper generalises the Euler-Maupertuis Principle of Least Action to systems with impact by formulating the principle as a variational inequality.

1 Introduction

In 1744 Euler stated that the motion of a point-mass m in potential field V is described by the variational principle

$$\sqrt{2m}\int_{x_0}^{x_1}\sqrt{E-V(x,y)}\sqrt{1+y'(x)^2}dx = \text{ stationary}, \qquad E = T+V = \text{const.},$$

where x and y are the coordinates of the point mass in an inertial frame. This principle is called the (Euler-Maupertuis) Principle of Least Action and is in this form sometimes known as the Principle of Jacobi. The aim of this paper is to put the Principle of Least Action in the context of non-smooth dynamics, giving the principle a novel application field.

2 Principle of d'Alembert-Lagrange as variational inequality

Let us consider a point-mass m which we can push with an external force $\boldsymbol{F}$ against a frictionless wall, or even into the corner of a wall. The wall reacts with a contact force $\boldsymbol{\lambda}$. Newton's second law

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{\lambda} \tag{1}$$

gives the equilibrium between the inertia force, the external force $\boldsymbol{F}$ and the contact force $\boldsymbol{\lambda}$. The contact force $\boldsymbol{\lambda}$ for a frictionless wall is an element of the normal cone,

$$-\boldsymbol{\lambda} \in \mathcal{N}_K(\boldsymbol{r}), \tag{2}$$

where the normal cone $\mathcal{N}_K(\boldsymbol{r})$ can be expressed as the subderivative $\partial\Psi_K(\boldsymbol{r})$ of the indicator function on the domain K which is enclosed by the wall. For simplicity we will assume K to be convex. If we substitute the equation of motion (1) in the set-valued force law (2), then we obtain a differential inclusion

$$-(m\ddot{\boldsymbol{r}} - \boldsymbol{F}) \in \partial\Psi_K(\boldsymbol{r}) \tag{3}$$

for the non-impulsive dynamics of the point-mass with unilateral constraint. Using some concepts of convex analysis we obtain the following result:

$$\begin{aligned} &\Psi_K(\boldsymbol{r}^*) \geq \Psi_K(\boldsymbol{r}) - \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{r}^* - \boldsymbol{r}) \quad \forall\, \boldsymbol{r}^* \\ &0 \geq -\boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{r}^* - \boldsymbol{r}) \quad \forall\, \boldsymbol{r}^* \in K, \quad \delta\boldsymbol{r} = \boldsymbol{r}^* - \boldsymbol{r} \\ &\boldsymbol{\lambda}^{\mathrm{T}}\delta\boldsymbol{r} = (m\ddot{\boldsymbol{r}} - \boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{r} \geq 0 \quad \forall\, \delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}) \end{aligned} \tag{4}$$

The virtual work $\boldsymbol{\lambda}^{\mathrm{T}}\delta\boldsymbol{r}$ of the contact force $\boldsymbol{\lambda}$ is nonnegative for all virtual displacements which are taken from the tangent cone $\mathcal{T}_K(\boldsymbol{r})$, which is the cone of all kinematically admissible virtual displacements. The expression $(m\ddot{\boldsymbol{r}} - \boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{r} \geq 0$, $\forall\, \delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r})$ is called a variational inequality.

Consider the principle of d'Alembert-Lagrange for a perfect bilateral constraint: the virtual work of a perfect bilateral constraint force vanishes for all kinematically admissible virtual displacements. The variational inequality which we just obtained is nothing else than the principle of d'Alembert-Lagrange for a perfect unilateral constraint.

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3 Principle of Hamilton in inequality form

We now integrate the principle of d'Alembert-Lagrange as variational inequality over a compact time-interval $I = [t_0, t_1]$. This time-interval can contain collision times which we gather in the set $D = \{t_i\}$. For the moment, we leave the impact times aside and we integrate over the interval $I\backslash D$:

$$\int_{I\backslash D} (m\ddot{\boldsymbol{r}} - \boldsymbol{F})^{\mathrm{T}} \delta\boldsymbol{r}\, \mathrm{d}t \geq 0 \qquad \forall\, \delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}). \tag{5}$$

As is customary in this kind of analysis we perform a partial integration.

$$\left[m\dot{\boldsymbol{r}}\delta\boldsymbol{r}\right]_{t_0}^{t_1} + \sum_i \left[m\dot{\boldsymbol{r}}\delta\boldsymbol{r}\right]_{t_i^-}^{t_i^+} - \int_{I\backslash D} m\dot{\boldsymbol{r}}^{\mathrm{T}}\delta\dot{\boldsymbol{r}}\, \mathrm{d}t - \int_{I\backslash D} \boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{r}\, \mathrm{d}t \geq 0$$

The boundary terms for t_0 and t_1 vanish for fixed boundary conditions $\boldsymbol{r}(t_0) = \boldsymbol{r}_0$, $\boldsymbol{r}(t_1) = \boldsymbol{r}_1$. The boundary terms at the collision times D vanish if the so-called Weierstrass-Erdmann conditions are fulfilled. The first Weierstrass-Erdmann condition requires that the impulsive forces lie within the normal cone, $-\boldsymbol{\Lambda} \in \mathcal{N}_K(\boldsymbol{r})$. The second Weierstrass-Erdmann condition demands that the collisions are completely elastic such that there is no energy loss during the impact. The second condition is superfluous if we consider a special class of test functions which lead to Gâteaux derivatives (see [1]). The remaining terms can be recognised as the variation of the kinetic energy $\delta T = m\dot{\boldsymbol{r}}^{\mathrm{T}}\delta\dot{\boldsymbol{r}}$ and the potential energy $\delta V = \boldsymbol{F}^{\mathrm{T}}\delta\boldsymbol{r}$, i.e.

$$-\int_{I\backslash D} \delta T\, \mathrm{d}t + \int_{I\backslash D} \delta V\, \mathrm{d}t \geq 0. \tag{6}$$

We are still integrating over the time-interval without collision times. However, the missing integration parts are Lebesgue integrals and the collision times are Lebesgue negligible:

$$\int_D \delta T\, \mathrm{d}t = 0 \qquad \int_D \delta V\, \mathrm{d}t = 0. \tag{7}$$

We obtain the principle of Hamilton as variational inequality,

$$-\delta \int_I L\, \mathrm{d}t \geq 0 \qquad L := T - V \quad \forall\, \delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}), \tag{8}$$

which also accounts for perfect unilateral constraints (see [1]).

4 Principle of Least Action in inequality form

In the same way we are also able to generalise the principle of least action to unilateral constraints. We can do so by starting from the principle of Hamilton (8) in inequality form and by taking the conservation of energy $E = T + V$ as auxiliary condition. This relates the variation of the potential energy $\delta V = -\delta T$ to the variation of the kinetic energy. Substitution of the latter in the principle of Hamilton gives a Lagrange function which is 2 times the kinetic energy

$$-2 \int \delta T\, \mathrm{d}t \geq 0 \quad \forall\, \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}),$$

in which we used generalised coordinates $\boldsymbol{q}$. A reformulation of the kinetic energy

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q}'^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}' \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$

gives the principle of least action in inequality form:

$$-\sqrt{2}\,\delta \int \sqrt{E_0 - V(\boldsymbol{q})}\sqrt{\boldsymbol{q}'^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'}\, \mathrm{d}\theta \geq 0 \quad \forall\, \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}).$$

Once again, we obtain a variational inequality.

References

[1] R. I. Leine, U. Aeberhard, and Ch. Glocker. "Hamilton's principle as variational inequality for mechanical systems with impact", to be submitted, 2008.