SOLVING NORMAL CONE INCLUSION PROBLEMS IN CONTACT MECHANICS BY ITERATIVE METHODS

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ABSTRACT

In this paper we show how inclusions originating from set-valued force laws can be written as non-linear equations for finite-dimensional second order dynamical systems. We express the set-valued force laws as subdifferentials of the indicator function to a given convex set $C$, apply the augmented Lagrangian and arrive at a proximal point problem which is solved by Jacobi or Gauss Seidel like iterative schemes. This proceed provides a simple unified access to frictional contact problems in dynamics and can be used either in event-driven or time-stepping simulation approaches.

1 INTRODUCTION

A point of mass in contact with a surface is a simple example of a non-smooth mechanical system. In order to solve the unilateral contact we must determine whether the point of mass stays further in contact with the surface or is separating. In the first case we have to calculate the positive normal contact force, in the second case we must determine the relative contact acceleration in normal direction. In the presence of friction we have to decide whether slip or stick is active. More generally, we have to deduce which description of the system is valid: opening contact, closed contact with stick active, closed contact with slip active. Our simple example is of course easy to solve, but considering more complex systems with several contacts, a mathematical framework is needed to solve the problem satisfactorily. Non-smooth contacts like unilateral and frictional contacts can be characterized by maximal monotone set-valued maps [Glocker(2001)]. For associated friction, those constitutive laws may even be formulated as normal cone inclusion, which relate them to optimization theory through the subdifferential of the indicator function. In both, event-driven [Glocker(1995)] and time-stepping simulation methods [Glocker and Studer(2005), Jean(1999), Moreau(1988), Studer and Glocker(2005)], such normal cone inclusions occur. They are solved by transforming the inclusions into linear and nonlinear complementarity problems [Cottle and Pang(1992), Glocker(1995), Glocker and Studer(2005)] or by using an augmented Lagrangian approach [Alart and Curnier(1991), Bertsekas(1982), Leine and Nijmeijer(2004), Studer and Glocker(2005)]. In this paper we restrict ourselves to the augmented Lagrangian approach. The entire work is based on the observation that normal cone inclusions can be stated as proximal point problems. The latter turn the associated inclusions into nonlinear equations, which provide the saddle-point conditions of the augmented Lagrangian function. A (cyclic) iterative solution of those equations yields a natural way Gauss-Seidel and Jacobi type iterative procedures with and without relaxation, which contain both, steps for solving linear equations but also conditional (nonlinear) projections, and which reduce to their classical versions when the normal cones are linear spaces. These algorithms provide a unified access to the numerical treatment of perfect unilateral and bilateral constraints, planar and spatial dry friction.

2 THE CONTACT PROBLEM ON ACCELERATION LEVEL

The purpose of this paper is to show how inclusions originating from set-valued force laws can be written as non-linear equations for finite-dimensional second order dynamical systems. To demonstrate the method, we have exemplarily chosen the dynamical take-off and stick-slip transition of frictional contacts within the event-driven approach. Impacts as well as contacts in pure sliding regime are intentionally left out for conciseness, but can be treated in a similar way. From the viewpoint of dynamics, we aim at determining the right accelerations of the system, i.e. the right derivatives $\ddot{u}(t)$ of the generalized velocities $u(t)$ for a given post-impact state $(q(t), u(t))$ at a fixed time $t$. The positions of the system at time $t$ are addressed by the generalized coordinates $q(t)$ with $\dot{q}(t) =$
We first review some basics of convex analysis in order to use this mathematical framework to describe the set-valued force laws [Glocker(1995), Rockafellar(1972)]. The indicator function $\Psi_C(x)$ of a convex set $C$ takes the value zero if $x \in C$ and infinity otherwise,

$$\Psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C. \end{cases}$$

The conjugate to this indicator function is the support function $\Psi_C^*(x^*)$,

$$\Psi_C^*(x^*) = \sup_x (x^T x^* | x \in C).$$

The subdifferential $\partial f(x)$ of a convex function $f(x)$ is a set containing the gradients of all supporting hyperplanes of $f(x)$,

$$\partial f(x) = \{ y | f(x^*) \geq f(x) + y^T (x^* - x); \forall x^* \}. \quad (3)$$

The subdifferential of the indicator function is identical with the normal cone $N_C(x)$, which is the set of all vectors making an obtuse angle with all other vectors emanating from $x \in C$ to any $\bar{x} \in C$,

$$N_C(x) = \{ y | y^T (\bar{x} - x) \leq 0; \forall \bar{x} \in C; x \in C \}. \quad (4)$$

Further, the subdifferential of the indicator function is the inverse of the subdifferential of the support function,

$$x^* \in \partial \Psi_C(x) \iff x \in \partial \Psi_C^*(x^*). \quad (5)$$

Non-smooth constraints can be described by set-valued force laws, which can be expressed as the subdifferentials of the indicator function to different convex sets $C_i$ [Glocker(1995), Glocker(2001)],

$$-\gamma_i \in \partial \Psi_C; (\lambda_i) \equiv N_C; (\lambda_i), \quad (6)$$

or using the conjugate

$$-\lambda_i \in \partial \Psi_C^*; (\gamma_i). \quad (7)$$

We denote by $\dot{\gamma}_i = W_i^T \dot{u} + \zeta_i$ the relative contact accelerations and by $\lambda_i$ the contact forces of the $i$-th constraint. The different non-smooth constraints differ from each other only in the sets $C_i$ of admissible contact forces, which we show exemplarily for the unilateral, spatial friction and bilateral constraint. A closed unilateral constraint is characterized by the set of all positive real numbers $\mathbb{R}_{+}$, and the set-valued force law becomes

$$-\dot{\gamma}_i \in N_{\mathbb{R}^+}; (\lambda_i) \quad \downarrow \quad \lambda_i > 0 \Rightarrow \dot{\gamma}_i = 0 \quad \lambda_i = 0 \Rightarrow \dot{\gamma}_i \geq 0.$$  

We recognize that the contact stays closed if there is a positive (pressure) contact force, otherwise it may open. A spatial dry friction constraint is defined by the circular constraint is characterized by the set of all real numbers $R^+$, and the set-valued contact forces of the $i$-th constraint. The set-valued force law is

$$\dot{\gamma}_i \in N_{\mathbb{R}^+}; (\lambda_i) \quad \downarrow \quad \| \lambda_i \| = a_i \Rightarrow \dot{\gamma}_i \in -R^+ \frac{\lambda_i}{\| \lambda_i \|} \quad \| \lambda_i \| < a_i \Rightarrow \dot{\gamma}_i = 0.$$  

If $\lambda_i$ is from the interior of the set $S$, then stick holds. Otherwise the contact may slide. Finally, a bilateral constraint is defined by the set of all real numbers $\mathbb{R}$. The associated set-valued force law is

$$\dot{\gamma}_i \in N_{\mathbb{R}^+}; (\lambda_i) \quad \downarrow \quad \forall \lambda_i \Rightarrow \dot{\gamma}_i = 0,$$

thus the contact stays closed for any contact force $\lambda_i$. 

Figure 1: Indicator (a) and support function (b), subdifferential (c) and normal cone (d) [Glocker(1995)]
3 FORMULATIONS IN NON-SMOOTH DYNAMICS

The purpose of this section is to show different adequate formulations of a non-smooth mechanical system. Applying the principle of virtual work to a system of nodes $\mathbf{\xi}$ with mass $dm$ and force distribution $d\mathbf{F}$ containing all internal and external forces,

$$\int_S \partial \mathbf{\xi}^T (\dot{\mathbf{\xi}} dm - d\mathbf{F}) = 0,$$

we obtain the equations of motion

$$\mathbf{M}\ddot{\mathbf{u}} - \mathbf{h} - \sum_{i}^{n} \mathbf{W}_i \lambda_i = 0,$$

(12)

where we have taken into account that $\dot{\mathbf{\xi}}$ force directions $\mathbf{\gamma}$ internal and external forces,

$$\sum_{i}^{n} \mathbf{W}_i \partial \Psi_{C_i}^*(\gamma_i),$$

with $\mathbf{M} = \mathbf{M}(\mathbf{q})$ being the positive definite mass matrix and $\mathbf{h} = \mathbf{h}(\mathbf{q}, \mathbf{u})$ the vector of all external and gyroscopic forces acting on the system. All constraints whose virtual work does not vanish are considered by the terms $\mathbf{W}_i \lambda_i$, in which $\mathbf{W}_i$ are the generalized force directions and $\lambda_i$ the contact forces of the $i$-th constraint. Elimination of the contact forces $\lambda_i$ in the equations of motion (12) yields

$$\mathbf{M}\ddot{\mathbf{u}} + \sum_{i}^{n} \mathbf{W}_i \partial \Psi_{C_i}^*(\gamma_i) \succeq 0,$$

(13)

whose integrated form is the associated optimization problem known as extended principle of least constraints [Glocker(2001)],

$$\min_{\mathbf{u}} (\mathbf{u} - \mathbf{u}_h)^T \mathbf{M}(\mathbf{u} - \mathbf{u}_h) + \sum_{i}^{n} \Psi_{C_i}^*(\gamma_i),$$

(14)

in which $\mathbf{u}_h = \mathbf{M}^{-1}\mathbf{h}$ are the unconstrained accelerations of the system. Instead of using the equations of motions (12), we can also transform the whole problem into contact coordinates to arrive at

$$\dot{\mathbf{\gamma}} = \mathbf{W}^T \mathbf{M}^{-1} \mathbf{W} \mathbf{\lambda} + \mathbf{W}^T \mathbf{M}^{-1} \mathbf{h} + \mathbf{\zeta} =: \mathbf{G}\mathbf{\lambda} + \mathbf{c},$$

$$-\dot{\gamma}_i \in \partial \Psi_{C_i}(\lambda_i),$$

$$\dot{\mathbf{\gamma}} = (\dot{\gamma}_1 \ldots \dot{\gamma}_n)^T, \quad \mathbf{\lambda} = (\lambda_1 \ldots \lambda_n)^T,$$

$$\mathbf{W} = (\mathbf{W}_1 \ldots \mathbf{W}_n), \quad \mathbf{\zeta} = (\zeta_1 \ldots \zeta_n)^T,$$

(15)

where we have taken into account that $\dot{\gamma}_i = \mathbf{W}_i^T \dot{\mathbf{u}} + \zeta_i$. Note that the matrix $\mathbf{G}$ is positive definite if the force directions $\mathbf{W}_i$ are independent, otherwise positive semidefinite. Elimination of $\dot{\mathbf{\gamma}}$ in (15) yields

$$\left( \begin{array}{c} \sum_{j=1}^{n} \mathbf{G}_{ij} \lambda_j + c_1 + \partial \Psi_{C_1}^*(\lambda_1) \\ \vdots \\ \sum_{j=1}^{n} \mathbf{G}_{nj} \lambda_j + c_n + \partial \Psi_{C_n}^*(\lambda_n) \end{array} \right) \succeq 0,$$

(16)

and the associated optimization problem is

$$\min_{\mathbf{\lambda}} \mathbf{\lambda}^T \mathbf{G}\mathbf{\lambda} + \mathbf{\lambda}^T \mathbf{c} + \sum_{i}^{n} \Psi_{C_i}(\lambda_i),$$

(17)

which is the dual optimization problem to the extended principle of least constraints (14). Because $\Psi_{C_i}(\lambda_i) = \infty$ for $\lambda_i \notin C_i$, the optimization problem (17) can also be stated as

$$\min_{\mathbf{\lambda}} \mathbf{\lambda}^T \mathbf{G}\mathbf{\lambda} + \mathbf{\lambda}^T \mathbf{c} \quad \text{for} \quad \lambda_i \in C_i.$$

(18)

4 AUGMENTED LAGRANGIAN

Constrained optimization problems of strictly convex cost functions $f(\mathbf{x})$ can be solved by an augmented Lagrangian approach. Regarding an optimization problem with affine equality constraints $g(\mathbf{x}) = 0$,

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{for} \quad g(\mathbf{x}) = 0,$$

(19)

the augmented Lagrangian function is [Bertsekas(1982)]

$$L_a(\mathbf{x}, \mathbf{\mu}) = f(\mathbf{x}) + \mathbf{\mu}^T g(\mathbf{x}) + \frac{r^2}{2} ||g(\mathbf{x})||^2.$$

(20)

The vector $\mathbf{\mu}$ are the Lagrange multipliers known from the ordinary Lagrangian, and $r^2$ an additional penalty parameter. Finding the saddle point of the augmented Lagrangian function, i.e.

$$\min_{\mathbf{x}} \max_{\mathbf{\mu}} L_a(\mathbf{x}, \mathbf{\mu}),$$

(21)

is equivalent to solving the optimization problem (19). In this section we show an elegant way how to derive the augmented Lagrangian function for an optimization problem of the form

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{for} \quad g(\mathbf{x}) \in C.$$

(22)

First we define a proximal point and a distance vector function: The proximal point function $\text{prox}_{C}(\cdot)$ performs a projection onto the set $C$, that is the point $\mathbf{\xi} = \text{prox}_{C}(\mathbf{x})$ is the nearest point to $\mathbf{x}$ in the set $C$, thus

$$\text{prox}_{C}(\mathbf{x}) = \text{argmin}_{\mathbf{\xi}} ||\mathbf{x} - \mathbf{\xi}|| \quad \forall \mathbf{\xi} \in C.$$

(23)

The distance vector function $\text{vdst}_{C}(\cdot)$ returns the shortest vector of the set $C$ to an arbitrary point $\mathbf{x}$,

$$\text{vdst}_{C}(\mathbf{x}) = \mathbf{x} - \text{prox}_{C}(\mathbf{x}).$$

(24)

We now replace the restriction $g(\mathbf{x}) \in C$ in (22) by the restrictions $g(\mathbf{x}) - \mathbf{\nu} = 0$ and $\mathbf{\nu} \in C$, and look for the
saddle point \( \min \max_{x, \nu \in C} L_a(x, \mu, \nu) \) of the augmented Lagrangian function

\[
L_a(x, \mu, \nu) = f(x) + \mu^\top (g(x) - \nu) + \frac{r^*}{2} ||g(x) - \nu||^2.
\]

(25)

Completing the square yields

\[
L_a(x, \mu, \nu) = f(x) - \frac{\mu^\top \mu}{2r^*} + \frac{r^*}{2} ||g(x) - \nu||^2. \tag{26}
\]

The augmented Lagrangian function \( L_a(x, \mu, \nu) \) is minimal with respect to \( \nu \in C \) if

\[

\nu_0 = \arg\min_{\nu \in C} \left( ||\frac{\mu}{r^*} + g(x)||^2 \right) = \text{prox}_{C}(\frac{\mu}{r^*} + g(x)).
\]

Inserting \( \nu_0 \) in (26) yields the augmented Lagrangian function

\[
L_{a0}(x, \mu) = f(x) - \frac{\mu^\top \mu}{2r^*} + \frac{r^*}{2} ||v_{	ext{dst}}C(\frac{\mu}{r^*} + g(x))||^2,
\]

(28)

which is continuously differentiable. The solution of the saddle problem \( \min \max L_{a0}(x, \mu) \) yields the conditions

\[

\frac{\partial L_{a0}}{\partial x} = \left( \frac{\partial f(x)}{\partial x} \right)^\top + r^* \frac{\partial g(x)}{\partial x} \text{v}_{\text{dst}}C(\frac{\mu}{r^*} + g(x)) = 0,
\]

(29)

\[

\frac{\partial L_{a0}}{\partial \mu} = - \frac{\mu}{r^*} + \text{v}_{\text{dst}}C(\frac{\mu}{r^*} + g(x)) = 0,
\]

(30)

which can also be written as

\[

0 = \left( \frac{\partial f(x)}{\partial x} \right)^\top + \frac{\partial g(x)}{\partial x} \mu - \frac{\mu}{r^*},
\]

\[

\mu = r^* \cdot \text{v}_{\text{dst}}C(\frac{\mu}{r^*} + g(x)). \tag{31}
\]

Equation (31) can be handled in two ways. A first possibility is to insert the definition of the function \( \text{v}_{\text{dst}}C(\cdot) \) to arrive at

\[

g(x) = \text{prox}_{C}(g(x) + \frac{\mu}{r^*}). \tag{32}
\]

The other possibility is to scale the distance problem by \( r^* \),

\[

\mu = r^* \cdot \text{v}_{\text{dst}}C(\frac{\mu}{r^*} + g(x)) = \text{v}_{\text{dst},*}C(\mu + r^*g(x)). \tag{33}
\]

When applying the augmented Lagrangian to our non-smooth mechanical system, we use the formulation \( (18) \) to obtain

\[

\dot{\gamma} = G\lambda + c, \tag{34}
\]

\[

-\ddot{\gamma}_i = r^* \cdot \text{v}_{\text{dst},C}(\frac{\cdot \dot{\gamma}_i}{r^*} + \lambda_i). \tag{35}
\]

in which the lagrangian multipliers \( \mu \) correspond to the negative contact accelerations \( \gamma_i \). Elimination of the contact accelerations \( \dot{\gamma} \) yields \( n \) equations for the \( n \) non-smooth constraints

\[

\lambda_i = \text{prox}_{C_i}(\lambda_i - r\sum_{i=1}^{n} G_{ij}\lambda_j + c_i)), \tag{36}
\]

which we call \( \lambda \)-equations. Note that we replaced in (36) the arbitrary positive scalar \( r^* \) by the arbitrary positive scalar \( \frac{1}{r^*} \). Of course it is also possible to eliminate the contact forces instead of the contact accelerations which yields the \( \dot{\gamma} \)-equations

\[

-\ddot{\gamma}_i = v_{\text{dst}*}C_i(-\ddot{\gamma}_i + r^*\sum_{i=1}^{n} G_{ij}^{-1}(\ddot{\gamma}_j - c_j)). \tag{37}
\]

The non-smooth mechanical system is well described now by either \( n \lambda \)-equations for the contact forces or \( n \dot{\gamma} \)-equations for the contact accelerations. A mechanical system with a non-regular matrix \( G \) causes problems. The \( \lambda \)-equations of such a system might have non-unique solutions, whereas a description of such a system by \( \dot{\gamma} \)-equations is not even possible. We will therefore only discuss the \( \lambda \)-equations in the following.

5 ITERATIVE SOLUTION TECHNIQUE

Regarding a system of one dimensional constraints \( (\lambda_i \in \mathbb{R}, C_i \subset \mathbb{R}) \), we can choose \( r = \frac{1}{G_{ii}} \) to solve the associated \( \lambda \)-equation of the \( i \)-th constraint straightforward as function of all other constraint forces \( \lambda_j \),

\[

\lambda_i = \text{prox}_{C_i}(\frac{1}{G_{ii}}\sum_{j=1}^{n} G_{ij}\lambda_j + c_i)). \tag{38}
\]

The fraction \( \frac{G_{ii}}{G_{ii}} \) defines the influence of the \( j \)-th constraint onto the \( i \)-th constraint. A possible iterative instruction to solve the problem is

\[

\lambda_i^{ \nu+1} = - \frac{1}{G_{ii}}\sum_{j=1}^{n} G_{ij}\lambda_j^{ \nu} + c_i), \tag{39}
\]

\[

\lambda_i^{ \nu+1} = \text{prox}_{C_i}(\lambda_i^{ \nu+1}).
\]

We assume that we know all contact forces \( \lambda_j^\nu \). Based on these \( \lambda_j^\nu \) we calculate the new contact forces \( \lambda_j^{\nu+1} \), neglecting the fact that already updated contact forces \( \lambda_j^{\nu+1} \) exist. An iterative instruction corresponding to a successive solution of the constraints [Moreau(1988)]
uses these already updated contact forces \( \lambda_{j \neq i}^{\nu + 1} \),
\[
\hat{\lambda}_{i}^{\nu + 1} = \frac{1}{G_{ii}} \left( \sum_{j \neq i}^{j \neq i} G_{ij} \lambda_{j}^{\nu + 1} + \sum_{j = i + 1}^{n} G_{ij} \lambda_{j}^{\nu} + c_{i} \right),
\]
(40)
\[
\lambda_{i}^{\nu + 1} = \text{prox}_{C_{i}}(\hat{\lambda}_{i}^{\nu + 1}).
\]

We recognize in instruction (39) the Jacobi and in (40) the Gauss Seidel iterative method for solving a linear system \( G \lambda + c = 0 \) combined with a projection. In other words, we solve an underlying linear system \( G \lambda + c = 0 \) by a Jacobi or Gauss Seidel method and perform projections onto the result in order to fulfill our set-valued contact laws.

Higher dimensional constraints \( (\lambda_{i} \in \mathbb{R}^{m}, C_{i} \subset \mathbb{R}^{m}) \) can only be solved straightforward as functions of all other constraint forces if the factor \( r \) can be chosen such that \( I - rG_{ii} = 0 \), which is in general not the case. Nevertheless, iterative instructions
\[
\lambda_{i}^{\nu + 1} = \lambda_{i}^{\nu} - r_{i} \left( \sum_{j = 1}^{n} G_{ij} \lambda_{j}^{\nu} + c_{i} \right),
\]
(41)
\[
\lambda_{i}^{\nu + 1} = \text{prox}_{C_{i}}(\hat{\lambda}_{i}^{\nu + 1}),
\]
or
\[
\lambda_{i}^{\nu + 1} = \lambda_{i}^{\nu} - r_{i} \left( \sum_{j = 1}^{j \neq i} G_{ij} \lambda_{j}^{\nu + 1} + \sum_{j = i + 1}^{n} G_{ij} \lambda_{j}^{\nu} + c_{i} \right),
\]
(42)
\[
\lambda_{i}^{\nu + 1} = \text{prox}_{C_{i}}(\hat{\lambda}_{i}^{\nu + 1}),
\]
can be applied. We recognize in (41) a Jacobi relaxation method (JOR) combined with a projection, in (42) a Gauss Seidel method (SOR) combined with a projection. Therefore we will call these two instructions the JORprox and SORprox method, respectively. As in the previous case we solve an underlying linear system \( G \lambda + c = 0 \) by a Jacobi or Gauss Seidel relaxation method and add a projection to fulfill the constraint conditions. Please note that convergence for the JOR method can only be guaranteed for strictly diagonal dominant matrices \( G \) and for the SOR method only for positive definite matrices \( G \) respectively. This is the case if the generalized force directions \( W_{i} \) are independent. If the constraints can act in a way that the system becomes underdetermined, the matrix \( G \) is only positive semidefinite and we must expect non-unique solutions for some contact configurations. The factor \( r_{i} \) corresponds to the relaxation factor which defines the influence of the other constraints on the \( i \)-th constraint. Generally, \( r_{i} \) has to be chosen “small” to guarantee convergence. On the other hand, a too small \( r_{i} \) conflicts with the termination criterion of the iteration. We suggest the following choice of parameter \( r_{i} \): If the matrix \( G \) is strictly diagonal dominant or positive definite, we set
\[
\frac{1}{r_{i}} = \frac{1}{G_{hh}},
\]
(43)
in which \( h \) denotes the row in \( G \) which belongs to constraint \( i \). In the case of more-dimensional constraints \( (\lambda_{i} \in \mathbb{R}^{m}, C_{i} \subset \mathbb{R}^{m}) \), \( h \) addresses any of the associated rows in \( G \), preferably the one which minimizes \( r_{i} \). If \( G \) is not strictly diagonal dominant or positive definite, then \( r_{i} \) has to be chosen small. A good empiric criterion is
\[
\frac{1}{r_{i}} = \sum_{k = 1}^{m} |G_{hk}|.
\]
(44)
If the diagonal elements predominate, then the criteria (43) and (44) become similar.

6 EXAMPLES

6.1 Block on surface

In this example we treat a planar system, consisting of a block on a surface with external loads \( F_{X}, F_{Y}, M \). The contact between the block and the surface is modeled by two unilateral and two planar frictional constraints, which act on the block’s lower corners (figure 2). We assume the unilateral constraints to be closed. For such a case, the equations of motion (12) are
\[
\begin{pmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J_{s}
\end{pmatrix} \begin{pmatrix}
x \dot{x} \\
y \dot{y} \\
j_{s} \dot{\xi}
\end{pmatrix} - \begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\ell & h & h & 0 & 0
\end{pmatrix} \begin{pmatrix}
\lambda_{1} & \lambda_{2} \\
\lambda_{3} & \lambda_{4} \\
\lambda_{5} & \lambda_{6}
\end{pmatrix} - \begin{pmatrix}
F_{x} \\
F_{y} \\
M
\end{pmatrix} = 0.
\]
We choose \( m = 2.1, J_{s} = 0.018, \ell = 0.15 \) and \( h = 0.05 \).
for which the \(\lambda\)-equations (36) become
\[
\lambda_1 = \text{prox}_{\frac{r_1}{G}}(\lambda_1 - r_1(1.8\lambda_1 - 0.8\lambda_2 - 0.4\lambda_3 - 0.4\lambda_4 + c_1)),
\]
\[
\lambda_2 = \text{prox}_{\frac{r_2}{G}}(\lambda_2 - r_2(-0.8\lambda_1 + 1.8\lambda_2 + 0.4\lambda_3 + 0.4\lambda_4 + c_2)),
\]
\[
\lambda_3 = \text{prox}_{\frac{r_3}{M}}(\lambda_3 - r_3(-0.4\lambda_1 + 0.4\lambda_2 + 0.6\lambda_3 + 0.6\lambda_4 + c_3)),
\]
\[
\lambda_4 = \text{prox}_{\frac{r_4}{S}}(\lambda_4 - r_4(-0.4\lambda_1 + 0.4\lambda_2 + 0.6\lambda_3 + 0.6\lambda_4 + c_4)),
\]
where \(c_i\) contains the terms \(c = W^TM^{-1}h\). The matrix \(G = W^TM^{-1}W\) is neither strictly diagonal dominant nor positive definite, but only positive semidefinite. Thus, we cannot guarantee convergence, and the system might have non-unique solutions for the forces \(\lambda\).

As an example, consider the case that the block remains in the stick state, which can be realized by moderate or vanishing loading \((F_X, F_Y, M)\). For this situation one can show that non of the \(\text{prox}_C\)-functions project on their sets \((\text{prox}_C(x) \equiv x)\). As a consequence, the \(\lambda\)-equations reduce to a set of \(\lambda\)-linear equations which are not independent due to the semi-definiteness of \(G\), and the contact forces can not be determined uniquely. However, we are already satisfied if the algorithm returns just one of those solutions, because in this example all possible force distributions lead to the same (unique) generalized accelerations \(\dot{u}\). If one of the contacts is going to slide or open, then the contact forces are expected to be unique. In these situations, the solution \(\lambda\) of the contact problem is at a point at which some of the \(\text{prox}_C\)-functions really project, which makes the \(\lambda\)-equations of the system independent.

The non-linear equations are solved by the JORprox and the SORprox method. In figure 3 the number of iteration steps is shown as a function of \(\frac{t}{t_E}\). Convergence cannot be guaranteed, but the iteration converges anyway for some \(r_i\). In the first plot both unilateral constraints are closed and both friction constraints stick. The solution is non-unique. In the plot we see the number of iteration steps of the JORprox method (dashed) and of the SORprox method (solid). In the second plot both unilateral constraints are closed and both friction constraints slide. Further, in the third plot we have one closed unilateral constraint. The corresponding friction constraint sticks. The other unilateral constraint opens.

6.2 Time Stepping

A discrete scheme to simulate non-smooth mechanical systems is Moreau’s Time stepping method [Glocker and Studer(2005), Leine and Nijmeijer(2004), Moreau(1988), Studer and Glocker(2005)]. The idea is to calculate velocity updates instead of accelerations and to use the integrated contact forces \(\Lambda_i = \frac{f^{E}_I}{\dot{\xi}_i} = t_{\dot{\xi}_i} \Lambda dt\) instead of the contact forces \(\dot{\Lambda}_i\). The instructions for a time step \(t_E = t_B + \Delta t\) are
\[
M(u_E - u_B) = h\Delta t + \sum_{i=1}^n W_i A_i, \quad y_E = y_B + u_B \Delta t,
\]
where \(\gamma_\epsilon = W_i^T u_E + \gamma_B = W_i u_B\). The \(\lambda\)-equations of this system can be found to be
\[
\Lambda_i = \text{prox}_{\tilde{C}_i}(\Lambda_i - \sum_{i=1}^n G_{ij} A_j + c_i),
\]
with
\[
G = W^T_M M^{-1}_M W_M, \quad c = (I + \epsilon)\gamma_B + W^T_M M^{-1}_M h M \Delta t.
\]

In order to simulate a system with thousand disks under frictional contacts, we consider only the contacts which are closed or penetrated at the midpoint \(q_M\) and characterize them by the set
\[
C_i = \{\xi \in \mathbb{R}^2 | \xi_1 \in \mathbb{R}_0^+, |\xi_2| < \mu \xi_1\}.
\]

The prox function to this set is
\[
(y_1, y_2) = \text{prox}_{C_i}(x_1, x_2) = \begin{cases} y_1 = x_1 & \text{if } x_1 \geq 0, \\ y_1 = 0 & \text{if } x_1 < 0, \\ y_2 = x_2 & \text{if } |x_2| \leq \mu y_1, \\ y_2 = \frac{x_2}{|x_2|} \mu y_1 & \text{if } |x_2| > \mu y_1. \end{cases}
\]

Figure (4) shows some snapshots of such a simulation.
Figure 4: Thousand balls falling in a funnel. Simulation of about half a million possible unilateral and frictional contacts with Moreau’s time stepping scheme. The restitution and friction coefficient are: \(\varepsilon_N = 0.3\), \(\varepsilon_T = 0\), \(\mu = 0.3\).

7 CONCLUSIONS

In this paper we showed how inclusions originating from set valued force laws can be written as non-linear equations using an augmented Lagrangian approach. We solved these non-linear equations iteratively by the JORprox or SORprox methods, which contain both, steps for solving linear equations but also conditional (nonlinear) projections. Special attention has to be paid when choosing the factor \(r\), which influences the convergence of the iterative methods. Finally we applied the theory on examples within the event-driven or time-stepping simulation approaches.

References


