THEORETICAL AND EXPERIMENTAL TREATMENT OF PERFECT
MULTI-CONTACT-COLLISIONS

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ABSTRACT

In this talk, we determine the region of feasible post-impact states for a finite degree of freedom multi-body system in dependence of the known pre-impact state in a geometrical way. We model the perfect contacts by means of set-valued force laws on displacement level. These force laws define a translated convex cone in the tangent space specifying the kinetically feasible post-impact states.

In addition to the kinetic and kinematic restrictions, the kinetic energy of an ideal constrained mechanical system is bounded with respect to what we call a reference velocity. Since some certain rheonomic multi-contact systems do not have reference velocities we are enforced to restate the energy concept applying the contact work concept. The goal of our approach is to end up with a general impact law by stating a parametrization of the set of all possible post-impact states.

In the second part of our talk, we validate the general impact law based on the equations mentioned above with experiments on multi-contact collisions. Tested configurations are chains of balls (Newton’s cradle). Configurations with chains of longitudinal bars and forked bars are scheduled. Based on the experimental data the post-impact states are discussed.

1 INTRODUCTION

In order to solve an impact due to a collision, an impact law is needed to fully determine the post-impact velocities. Impact laws are commonly expressed in terms of restitution coefficients. The latter can be defined on kinematic (Newton), on impulsive (Poisson) (Glocker, 2001a) or an energetic (Stronge et al., 2001) level. Single-contact-collisions can be treated in such a way. However, serious problems arise if many contacts are closed at the time of impact. The impact dynamics of such problems gets global and cannot be correctly described by local restitution coefficients. A simple extension of Newton’s impact law to multi-contact scenarios may give incorrect results. Even if Newton’s impact law is extended such that it respects possible interferences of the contacts, not all of the possible post-impact states may be addressable. Useful parameters for multi-contact collisions are the impulse correlation ratios (ICR) (Ceanga and Hurmuzlu, 2001) and the Frémond matrix (Frémond, 1995; Glocker, 2001a). While the ICR is again an impulse estimate, the Frémond matrix is a purely kinematic approach. This paper gives a general approach how to handle multi-contact collisions in a geometrical way. The systems we treat are perfectly constrained multi-body-systems with finite degree of freedom with kinetical and kinematical excitation.

One of the problems in impact dynamics is that the term “impact” is often misunderstood. Many authors call their work impact dynamics, but they merely solve fast collision dynamics. An impact is by definition a velocity jump, which is accompanied by impulsive forces. Impacts can be realized in different ways, e.g. by choosing for the system a stiffness distribution c and letting it tend to infinity. The fact that the post-impact velocities depend on the choice of the stiffness distribution is an integral component of impacts and not something that must be prevented. If this fact is not accepted, it may lead to senseless discussions when differences for “identical” problems occur. For the treatment of impacts in the spirit of the above definition we refer in particular.
2 THEORY

We consider a classical mechanical system with perfect scleronomic bilateral constraints. Such a system is geometrically represented by a finite-dimensional Riemannian manifold \((\mathcal{M}, \langle \cdot, \cdot \rangle)\) with associated kinetic metric that induces an inner product \(\langle \cdot, \cdot \rangle\) on the tangent bundle. Let \(I\) be an open real interval, containing the time instance \(t\) of a possible collision. For \(q \in \mathcal{M}\), let \(T_q(\mathcal{M})\) be the tangent space to \(\mathcal{M}\) at \(q\). An element of \(T_q(\mathcal{M})\) is therefore a derivative on \(C^1(\mathcal{M}, \mathbb{R})\) and represents possible velocities and virtual displacements which the system may have at this point \(q\). Disregarding nonsmooth effects like collisions, the trajectory of the system is \(C^2\) in time. The equation of motion is

\[
\frac{d}{dt} \mathbf{u} = \mathbf{f}(t, q), \quad \mathbf{u} = \dot{q} 
\]

where \(\frac{d}{dt}\) is the covariant derivative and \(\mathbf{f}\) is a 1-form on \(\mathcal{M}\). It represents the total forces acting on the system. The operator \(\sharp\) is the canonical isomorphism induced by the inner product from the cotangent space to the tangent space and assigns "accelerations" to forces. It represents therefore the inverse inertia of the system.

We define a set of real \(C^1\)-functions on space-time with non-vanishing sets of zeros and non-vanishing spatial differentials on them,

\[
\mathcal{G} := \{ g \in C^1(I \times \mathcal{M}, \mathbb{R}) \mid \\
\forall t \in I : g_t^{-1}(0) \neq \emptyset, \\
\forall q \in g_t^{-1}(0) : dg_{t,q} \neq 0 \}
\]

with \(g_t(q) := g(t, q)|_{t=\text{const.}}\). The assumption \(dg_t \neq 0\) is some sort of "constraint qualification assumption", (Aubin and Frankowska, 1990) which is necessary to assign a force direction to every closed contact. Contact phenomena for which this assumption does not hold are, among others, non-regularities in the boundary of the set of admissible displacements like reentrant corners (Glocker, 2002), but also pathological cases when the bodies consist only of boundaries like a pair of mass points. An entity \(q \in \mathcal{G}\) describes locally the signed normal distance (gap, contact, penetration) of two bodies with \(C^1\)-boundaries and represents a possible contact.

We remark that contact detection is a difficult task and just claim that the "interesting" contacts are already known. These are the contacts which are in the neighborhood of zero, i.e. nearly closed or fully closed. It follows from the definition that non-regular points like reentrant corners are not treated in this paper, as well as \(C^0\)-functions with respect to space (Glocker, 1999) or time.

We collect all relevant independent contacts of the mechanical model in the set \(G\) and assume \(G\) to be finite. A displacement \(q\) is called admissible with respect to \(G\), if \(\forall i \in G : g_i(t, q) \geq 0\). Let \(C_t\) be the set of all admissible displacements at a fixed time \(t\),

\[
C_t := \{ q \in \mathcal{M} \mid g_i(t, q) \geq 0 \forall i \in G \}.
\]

For all \(t \in I\), we postulate that \(C_t\) is not empty. As we assumed the constraints to be "hard", jumps in velocities, i.e. first derivatives of the displacements must be allowed. We therefore choose the motion \(t \mapsto q\) to be absolutely continuous, and allow the velocities \(t \mapsto \mathbf{u}\) to jump at an at most countable subset \(J\) of \(I\). We assume \(t\) not to be an accumulation point of \(J\) to prevent us from further difficulties. For every \(t\), a left- and right-velocity \(\mathbf{u}^-, \mathbf{u}^+\) as the pre- and post-impact velocity is defined.

If accelerations are not defined anymore, i.e. if \(t \in J\), equation (1) becomes meaningless. Hence we extend (1) in the following way: We replace \(t \mapsto \mathbf{f}(t, q(t))\) by a primitive function \(\mathbf{F}\), i.e. \(\mathbf{f} = \dot{\mathbf{F}}, (t \notin J)\) which is a posteriori known, and replace the derivatives of the functions with bounded variation \(\mathbf{u}(t), \mathbf{F}(t)\) by their Stieltjes-differential-measures (Moreau, 1988)

\[
\mathbf{Du} = 2d\mathbf{F}.
\]

Since the Lebesgue-density of a Stieltjes-measure of a \(C^1\)-function is the function itself, equation (1) may then be interpreted as a relation between densities of measures with respect to the Lebesgue-measure \(d\mathbf{F}\). If we take densities with respect to the Dirac-measure \(\delta_\mathcal{H}\), we get the impact equation:

\[
\mathbf{u}^+ - \mathbf{u}^- = 2\mathcal{H}\mathcal{F}
\]

with \(\mathcal{F} := \mathcal{F}^+ - \mathcal{F}^-\) as the total generalized impulse acting on the system. Equation (5) correlates the velocity jump to the total impulse acting on the system. Assuming an admissible displacement \(q\), the contacts which are closed shall be gathered into the set \(\mathcal{H}_t(q)\),

\[
\mathcal{H}_t(q) := \{ i \in G \mid g_i(t, q) = 0 \}.
\]

The virtual relative displacements \(\delta g^i\) are defined by

\[
\delta g^i := dg^i(\delta q),
\]

where \(\delta q := \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} q(t)\) is the virtual displacement.
The left-relative velocities are defined as $\gamma^{-} := dq^i_t(u^-) + g^i$, and the right-relative velocities as $\gamma^{+} := dq^i_t(u^+) + g^i$, respectively. Further, the affine map $u^+ \mapsto \gamma$ can be extended to all test velocities $u^* \in T_M(q)$ at fixed $q$:

$$\gamma^*(u^*) := dq^i_t(u^*) + g^i. \quad (8)$$

The kinematic restrictions for the left- and right-relative velocities and the virtual relative displacements of the closed contacts are

$$\forall i \in H_t(q) : \gamma^i \leq 0 \leq \gamma^i, \quad 0 \leq \delta g^i \quad (9)$$

which prevents the contacts to penetrate for both time arrows, the physical and the inverse, but also for the virtual arrow. By using the affine and linear maps (7,8), these $3|H_t(q)|$ inequalities (9) define a convex cone $T_C(q)$ and two convex polyhedra $P_C^\pm(q)$ as subsets of the tangent space, which are the sets of all admissible generalized virtual displacements $\delta q$ and left- and right-velocities $u^-, u^+$ for a given $(t,q)$,

$$T_C(q) := \{\delta q \in T_M(q) \mid dq^i_t(\delta q) \geq 0 \ \forall i \in H_t(q)\},$$

$$P_C^+(q) := \{u \in T_M(q) \mid \gamma^i(u) \geq 0 \ \forall i \in H_t(q)\},$$

$$P_C^-(q) := \{u \in T_M(q) \mid \gamma^i(u) \leq 0 \ \forall i \in H_t(q)\}, \quad (10)$$

see figure 1. We set $T_C(q), P_C^\pm(q) := \emptyset$, if $q \notin C_t$. For all $t \in I$ and all $q \in C_t$, we postulate that $T_C(q), P_C^\pm(q)$ are not empty. We call $T_C(q)$ the tangent cone to the set $C_t$ at $q$ which is the set of all admissible virtual displacements, and $P_C^\pm(q)$ the left- and the right-admissibility ranges, which are the sets of all admissible left- or right-velocities. These sets degenerate to the full tangent space $T_M(q)$, if no contact is closed, that means, if $q \in \text{int}(C_t)$. Other important properties are: Corresponding to the fact that a relative velocity cannot have both signs at the same time unless it vanishes, the interiors of $P_C^-(q)$ and $P_C^+(q)$ do not intersect. Both polyhedrons have the same recession cone (up to the sign), which is the tangent cone $T_C(q)$, see Rockafellar (1972). Given a trajectory $t \mapsto q$ and a point in time $t \in I$, the trajectory is kinematically admissible in $q$ at time $t$, if $q \in C_t$ and $u^- \in P_C^-(q), u^+ \in P_C^+(q)$.

### 2.1 Contact model

We define perfect unilateral constraints by the following characteristics on the contact force $\lambda$ and contact impulse $\Lambda$: (i) it has no tangential component (i.e. it is frictionless), (ii) it vanishes if the contact is open, (iii) it is non-adhesive, (iv) the contact force is not bounded. Point (iv) is a consequence on the "hard" kinematics and finite spatial discretisation. The constitutive law of a perfect unilateral constraint on displacement level $(g \mapsto \lambda)$ is expressed by the set-valued map $-\lambda \in Upr(g)$ (Glocker, 2001b) or equivalently, written as a complementarity, by

$$0 \leq g \perp \lambda \geq 0. \quad (11)$$

We introduce the cone $N_C(q)$ as the set of all non-negative linear combinations of the generalized contact force directions $dg^i_t, \ i \in H_t(q)$:

$$N_C(q) := \{-\Lambda, dg^i_t \mid (\Lambda, i) \geq 0\} \quad (12)$$

as a subset of $T_C^*(q)$. In (12) we used the summation convention $\sum := \sum_{i \in H_t(q)} dq^i_t$, i.e. summation over the closed contacts is performed if an index appears twice, as an upper and a lower one. In contrast, an indexed term in parentheses addresses all elements for all closed contacts, $(\Lambda_i) := (\Lambda_i)_{i \in H_t(q)} \in \mathbb{R}^{|H_t(q)|}$. If $q \in \text{int} C_t$, i.e. $H_t(q) = \emptyset$, we define $N_C(q) = \{0\}$ and if $q \notin C_t$, we define $N_C(q) = \emptyset$. Further, we define $T_C^*$ as the dual of $N_C(q)$ with respect to the kinetic metric

$$T_C^*(q) := \mathbb{N}N_C(q), \quad (13)$$

and call $T_C^\perp(q)$ the orthogonal cone to $C_t$ at $q$. As a consequence of (11) (Aeberhard and Glocker, 2005), the generalized contact impulse $R \in T_C^*(q)$ is an element of the cone $-N_C(q)$:

$$\text{perfect constraints} \Leftrightarrow -R \in N_C(q). \quad (14)$$

Let $P \in T_C^*(q)$ be the impulse of an external force excitation with measure in time $dP$. The total impulse is then $F = R + P$. Together with (14), the impact equation (5) becomes an inclusion for the unknown $u^\pm$ which is called the impact inclusion

$$u^+ \in u^- + \mathbb{N}P - T_C^*(q), \quad (15)$$

see figure 1.

### 2.2 Contact work

Consider a path $q(t)$ with left- and right-relative velocities $(\gamma^{-})$ and $(\gamma^{+})$. Let further be $\check{\gamma} := \frac{1}{2}(\gamma^{+} + \gamma^{-})$. We define the contact work as

$$W_R = \Lambda \cdot \check{\gamma} \quad (16)$$

(Aeberhard and Glocker, 2005) where, again, summation has to be taken over all closed contacts $i \in H_t(q)$. The work of a single contact $i$ is $\Lambda_i \check{\gamma}^i$ (no summation). If the term is negative, then the contact consumes mechanical energy. If it is positive, the contact releases mechanical energy to the system.

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2.3 Kinetic energy and contact work

Let \( \mathbf{u} \) be an arbitrary velocity of the system and \( \mathbf{a} \) be the velocity of an observer. The kinetic energy of the system with respect to the observer is

\[
T_a(\mathbf{u}) := \frac{1}{2} \| \mathbf{u} - \mathbf{a} \|^2 ,
\]

where \( \| \cdot \| \) is the kinetic norm induced by the inner product \((\cdot, \cdot)\). The relation between the kinetic energy and the contact work is as follows (Aeberhard and Glöckler, 2005): The jump in the kinetic energy, \( T_a(\mathbf{u}^+) - T_a(\mathbf{u}^-) \), is the sum of the work of the kinematic excitation \( W_M = -\Lambda_i \gamma'(a) \) and the work of the kinetic excitation \( W_P = \mathcal{P}(\mathbf{u} - a) \).

\[
T_a(\mathbf{u}^+) - T_a(\mathbf{u}^-) = W_R + W_M + W_P .
\]

2.4 Contact work postulate

For systems without kinetic excitation (\( d\mathbf{P} = 0 \)) we postulate that the overall work done by the contacts is not positive,

\[
d\mathbf{P} = 0 \Rightarrow W_R \leq 0 .
\]

Note that a vanishing external impulse \( \mathcal{P} = 0 \) does not imply that the system is not kinetically excited, \( \mathcal{P} = 0 \not\Rightarrow d\mathbf{P} = 0 \). The contact work postulate does therefore not apply if external impulse just vanishes, as recognized by the example \( d\mathbf{P} := \lim_{\tau \to 0}(\delta t - \delta t') \), see Aeberhard and Glöckler (2005) for more details. By using the energy-work-relation (18), the contact work postulate (19), definition (10) and the contact characteristic (11), it follows that the jump of the kinetic energy with respect to an observer with velocity \( \mathbf{a} \in \mathcal{P}_C^+(q) \) is not positive,

\[
d\mathbf{P} = 0 \Rightarrow \forall \mathbf{a} \in \mathcal{P}_C^+(q) : T_a(\mathbf{u}^+) - T_a(\mathbf{u}^-) = W_R - \Lambda_i \gamma'(a) \leq 0 .
\]

With the help of (17), this statement can geometrically be interpreted: Sets with bounded kinetic energy \( \{ \mathbf{u} \mid T_a(\mathbf{u}) \leq \tau/2 \} \) are balls \( B_r(\mathbf{a}) \) (with respect to the kinetic metric) with radius \( r \) and midpoint \( \mathbf{a} \). An energetically admissible right-velocity \( \mathbf{u}^+ \) has to be an element of the ball \( B_r|\mathbf{u}^- - \mathbf{a}|(\mathbf{a}) \). Because \( \mathbf{a} \) is arbitrary in \( \mathcal{P}_C^+(q) \), inequality (20) has to hold for every \( \mathbf{a} \in \mathcal{P}_C^+(q) \):

\[
d\mathbf{P} = 0 \Rightarrow \mathbf{u}^+ \in \bigcap_{\mathbf{a} \in \mathcal{P}_C^+(q)} B_r|\mathbf{u}^- - \mathbf{a}|(\mathbf{a}) ,
\]

see figure 1. This generally infinite intersection may equivalently be replaced by the finite intersection over \( E^+ := \text{Extr} \left( \mathcal{P}_C^+(q) \cap \mathcal{L}(\mathcal{P}_C^+(q))^\perp \right) \) which is basically the set of all extreme points of some sub-polyhedron of \( \mathcal{P}_C^+(q) \), see Aeberhard and Glöckler (2005). By \( \mathcal{L}(C) \) we denote the lineality-space of a convex set \( C \), which is the intersection of the recession cones of \( C \) and \(-C \), see Rockafellar (1972).

Detection of \( E^+ \). Since extreme points are zero-dimensional faces, they are the intersections of \( m = \dim(\mathcal{L}(\mathcal{P}_C^+(q))) \) pairwise non-parallel affine hyperplanes \( \mathcal{P}^+ = \{ \mathbf{x} \in (\mathcal{L}(\mathcal{P}_C^+(q)))^\perp \mid \gamma'(\mathbf{x}) = 0 \} \) which have, in addition, to be kinematically admissible,

\[
E^+ = \{ \mathbf{a} \in \mathcal{P}_C^+(q) \cap \mathcal{L}(\mathcal{P}_C^+(q))^\perp \mid \exists H \subset H(q) : |H| = \dim(\mathcal{L}(\mathcal{P}_C^+(q)))^\perp , \forall i \in H : \gamma'(a_i) = 0 \} .
\]

For a given left-velocity \( \mathbf{u}^- \in \mathcal{P}_C^-(q) \) and given excitation \( d\mathbf{P} \), we determine now the set \( \mathcal{S} \subset T_M(q) \) of possible right-velocities. Using the equalities and inclusions (9,10,15,21), we get:

\[
\mathcal{S} = \mathcal{P}_C^+(q) \cap (\mathbf{u}^- - T_C(q)) = \bigcap_{\mathbf{a} \in E^+} B_{\|\mathbf{u}^- - \mathbf{a}\|}(\mathbf{a})
\]

\[
(d\mathbf{P} = 0),
\]

\[
\mathcal{S} \subset \mathcal{P}_C^+(q) \cap (\mathbf{u}^- + \varepsilon \mathcal{P} - T_C(q))
\]

\[
(d\mathbf{P} \) arbitrary).
\]

This set \( \mathcal{S} \) of possible right-velocities is depicted in figure 1 for a non-uniform kinematically excited system.

2.5 Properties of \( \mathcal{S} \)

For both cases, \( d\mathbf{P} = 0 \) and \( d\mathbf{P} \) arbitrary, there exists a unique admissible right-velocity \( \mathbf{u}_P \in \mathcal{S} \) with minimal distance to \( \mathbf{u}^- \) in the sense of the kinetic metric, see figure 1. Uniqueness is guaranteed by the convexity of \( \mathcal{P}_C^+(q) \). Existence is proven in (Aeberhard and Glöckler, 2005) by arguments from convex analysis which ensure that \( \mathcal{S} \) is never empty. For systems without external impulsive excitation, \( \mathbf{u}_P \) corresponds to the right-velocity that causes minimal contact work, i.e. maximal dissipation. \( \mathcal{S} \) is convex and bounded for systems without kinetic excitation. The energetic level \( \partial_{\mathbf{a} \in \mathcal{E}^+} B_{\|\mathbf{u}^- - \mathbf{a}\|}(\mathbf{a}) \) represents collision events that show globally a non-dissipative impact behavior, whereas \( \mathbf{u}_P \) uniquely addresses the most inelastic impact. Velocities with constant contact-impulse ratios \( \alpha_{ij} := \Lambda_i/\Lambda_j \), \( i, j \in \mathcal{H}_k(q) \) are those half-lines in (15) that emanate from \( \mathbf{u}^- \) (Ceanga and Hurmuzlu, 2001).

2.6 Special cases

Uniform kinematical excitation. We call a system uniformly excited if there exists an observer, for which all
relative velocities of all closed contacts vanish, \( \exists \mathbf{a} : \gamma^i(\mathbf{a}) = 0 \ \forall i \in \mathcal{H}_i(q) \). Such an observer velocity \( \mathbf{a} \) is called a reference velocity. In this case, the set \( \mathcal{P}_C(q) \) degenerates to the (translated) tangent cone \( \mathbf{a} + \mathcal{T}_C(q) \), and the set of extreme points \( E^+ \) in (21) reduces to a singleton which is the tip of the cone and therefore is a reference velocity, which is unique in this case. Therefore,

\[
S = (\mathbf{a} + \mathcal{T}_C(q)) \cap (\mathbf{u}^- - \mathcal{T}^+_C(q)) \\
\cap B_{||\mathbf{u}^- - \mathbf{a}||}(\mathbf{a}) \quad (d\mathbf{P} = 0),
\]

\[
S \subset (\mathbf{a} + \mathcal{T}_C(q)) \cap (\mathbf{u}^- + \mathbf{zP} - \mathcal{T}^+_C(q)) \quad (d\mathbf{P} \text{ arbitrary}).
\]

\[ S = A \cap B \cap C, \]

\[
A = \mathcal{P}_C(q) \\
B = \mathbf{u}^- - \mathcal{T}^+_C(q) \\
C = \bigcap_{a \in E^+} B_{||\mathbf{u}^- - \mathbf{a}||}(\mathbf{a}) \quad (d\mathbf{P} = 0).
\]

Therefore, any parametrization of \( A, B \) or \( C \) is also a parametrization of \( S \), if the parameter domain is accordingly restricted. In the case \( A \subset B \) (or \( B \subset A \)), parameters of the contained set \( A \) (or \( B \)) do not depend on the containing set \( B \) (or \( A \)), and a restriction of the parameter domain is only necessary with respect to \( C \). Natural parameters of \( A \) and \( B \) are contact parameters as the relative velocities (\( \gamma^i \)) or contact impulses (\( \Lambda_i \)).

### 2.8 Example: linear chain

We consider a linear chain of \( n \geq 2 \) rigid bodies with masses \( m_1, m_2, \ldots, m_n > 0 \) and denote the total mass by \( m := \sum_{i=1}^n m_i \). The configuration manifold is \( \mathcal{M} = \mathbb{R}^n \), and the kinetic metric tensor is \( \mathbf{M} = \text{diag}(m_1, m_2, \ldots, m_n) \).

The contacts are \( g^i(q) = q^{i+1} - q^i, i = 1, \ldots, n-1 \). We are only interested in \( q = 0 \), for which all bodies are in contact and the index set is full, \( \mathcal{H}_i(0) = \{1, 2, \ldots, n-1\} \). The system is scleronomic, and we have \( \mathcal{P}_C(0) = \mathcal{T}_C(0) \). No external excitation is considered. Therefore, the contact work postulate applies. We get

\[
dq_i^g = dq^{i+1} - dq^i, \quad i \in \mathcal{H}_i(0),
\]

\[
\mathcal{T}_C(0) = \{ \mathbf{u} \in \mathbb{R}^n \mid 0 \leq dq_i^g(\mathbf{u}) = u^{i+1} - u^i \quad \forall i \in \mathcal{H}_i(0) \}
\]

\[
= \{ \mathbf{u} \in \mathbb{R}^n \mid u^1 \leq u^2 \leq \cdots \leq u^n \},
\]

\[
\mathcal{T}^+_C(0) = \{ \sum_{i=1}^{n-1} A_i \left( \frac{e_i}{m_i} - \frac{e_{i+1}}{m_{i+1}} \right) \mid (A_i) \geq 0 \},
\]

\[
B_r(0) = \{ \mathbf{u} \in \mathbb{R}^n \mid \sum_{i=1}^n (m_i(u^i)^2) \leq r \},
\]

where \( e_i \) is the \( i \)-th canonical basis vector of \( \mathbb{R}^n \). By using the domains in (27), we get from (25)

\[
S = \mathcal{T}_C(q) \cap (\mathbf{u}^- - \mathcal{T}^+_C(q)) \cap B_{||\mathbf{u}^-||}(0)
\]

\[
= \{ \mathbf{u} \in \mathbb{R}^n \mid u^1 \leq \cdots \leq u^n, \}
\]

\[
\frac{1}{2} m_i(u^i)^2 \leq \frac{1}{2} m_i(u^-^i)^2 \}
\]

Figure 1: Velocities of a colliding system without kinetic excitation (\( d\mathbf{P} = 0 \), with not-uniform kinematic excitation.

\[ A \subset B \subset C \]

\[ S = A \cap B \cap C \]

\[ A = \mathcal{P}_C(q) \]

\[ B = \mathbf{u}^- - \mathcal{T}^+_C(q) \]

\[ C = \bigcap_{a \in E^+} B_{||\mathbf{u}^- - \mathbf{a}||}(\mathbf{a}) \quad (d\mathbf{P} = 0) \]

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Note that the lineality space $\mathcal{L}(P_{C_i}^+(0))$ is the diagonal $\mathbb{R}d$ of $\mathbb{R}^n$ with $d = e_1 + \cdots + e_n$. Let $u^+, u^- \in \mathbb{R}^n$ be a left- and right-velocity, and let $u^+_i, u^-_i$ be the mass-orthogonal projections of $u^+, u^-$ onto $\mathcal{L}(P_{C_i}^+(0))$. Because every single generalized contact force direction $dq_i^j$ is orthogonal to $\mathcal{L}(P_{C_i}^+(0))$, the impact inclusion (15) implies that the projections $u^+_i$ remain constant during the collision, $u^+_i = u^-_i$. The physical interpretation of this fact is the conservation of total linear momentum, because the chain is kinetically not excited. We obtain

$$u_s = u_s d, \quad u_s = \frac{1}{m} \sum_{i=1}^n m_i u^+_i,$$

where $u_s$ is the velocity of the center of mass of the chain. Therefore the velocity jumps $u^+ - u^-$ are exclusively elements of $(\mathcal{L}(P_{C_i}^+(0)))^\perp$, which suggest itself to parameterize this $(n-1)$-dimensional subspace of $\mathbb{R}^n$. There are two natural choices: (i) the relative velocities $(\gamma^i)$ or (ii) the contact impulses $(\Lambda_i)$. One can show\(^1\) that kinematical restrictions ($\pm(\gamma^i) \geq 0$) are always stronger than restrictions on the contact impulses ($\Lambda_i \geq 0$) ($A \subset B$ in (26)). A parametrization via the relative velocities $(\gamma^i)$ would therefore not have to obey any restrictions coming from the contact impulses. The only restrictions on these parameters – apart from the trivial ones $(\gamma^i) \geq 0$ – is the energy-restriction

$$T^+ \leq T^-$$

with

$$T_r(\gamma^i) = \frac{1}{2} (\gamma^i)^\top G(\gamma^i)$$

$$T = T_s + T_r,$$

$$T_s = \frac{1}{2} m \sum_{i=1}^n u^2_i = \text{const.}$$

The symmetric and regular matrix $G \in \mathbb{R}^{n-1,n-1}$ represents $M$ in contact coordinates $(g^i)$. A calculation yields $(G)_{ij} = \frac{(m_1 + \cdots + m_i)(m_j + \cdots + m_n)}{m}, \quad 1 \leq i \leq j \leq n$.

The parametrization $(\gamma^i) \leftrightarrow u \in \mathbb{R}^n$ has the form $u(\gamma^i) = u_s d + \gamma^i f_i$, $(\gamma^i) \geq 0$, $T^+ \leq T^-$ with $f_i = (\alpha_i - 1)(e_1 + \cdots + e_i) + \alpha_i (e_{i+1} + \cdots + e_n) \in \mathcal{L}(P_{C_i}^+(0))$, $\alpha_i := \frac{m_1 + \cdots + m_i}{m}, \quad i \in \mathcal{H}_i(0)$. The vectors $f_i$ generate $(\mathcal{H}_i(0)) \cap (\mathcal{L}(P_{C_i}^+(0)))^\perp$ by nonnegative linear combinations.

Relative velocities lead to the concept of the Frémond matrix $e \in \mathbb{R}^{n-1,n-1}$ which provides a linear map between the left- and right relative velocities,

$$(\gamma^+)^i = -e(\gamma^-)^i.$$  

The physical interpretation is non-locality of contacts: the off-diagonal elements describe the action between different contacts (e.g. non-neighboring bodies).

### 2.9 3-ball chain

In the following we consider a chain of three balls of equal masses $m$, see Glocker and Aeberhard (2006). The section $S$ of possible post-impact velocities is bounded by the kinematical constraint and the energy constraint,

$$S = \{u_s d + \gamma^+ f_1 + \gamma^2 f_2 + \gamma^3 f_3 | \gamma^1, \gamma^2, \gamma^3 \geq 0, \gamma^+ \geq 0, (\gamma^+)^2 + \gamma^1 + \gamma^2 + \gamma^3 \leq (\gamma^1)^2 + \gamma^1 - \gamma^2 - (\gamma^2)^2\},$$

Initially ball 1 has the velocity $u_1^- \geq 0$, and the other two balls are in contact and at rest. Hence, due to conservation of momentum, the feasible post impact velocities are restricted to the plane $u_1^+ + u_2^+ + u_3^+ = u_1^- = \text{const.}$, which is shown in figure 2. The generalized velocities $u$ are projected along the diagonal of the velocity space onto this plane. This diagonal is perpendicular to the plane of admissible post-impact states, $\mathcal{L}(P_{C_i}^+(0))^\perp$. Condition (9) restricts the post-impact velocities to one fourth of the velocity space. The intersection with the momentum plane provides the cone depicted in figure 2. This cone is limited by the sphere of conservation of the kinetic energy of the relative motion, cf. equation (30). In the center of the energy ball we identify the point of maximum dissipation $I$ which lies on the diagonal of the velocity space and corresponds to the state, where all three balls have the velocity $u_s$ of the center of mass after the impact.

### 3 EXPERIMENT

Experiments were made with chains of balls, $n = 2, 3, 4, 5$, (Pavy et al., 2005). The aim was to determine the Frémond matrix for left-velocities $u^1 = 25\text{mm/s}$, $u^i = 0, i > 1$.

#### 3.1 Setup

The basic setup consists of a base frame supported on a damped optical table. The damped table is necessary to isolate the setup from vibrations of the building, cf. figure 3. The base frame and custom-built mounting equipment assure precise hanging of the test bodies.
with a pendular radius of 700mm. As shown in figure 4 the tested bodies are mounted by use of pairs of parallel threads plaited with tempered silicone-PTFE (STROFT GTP fishing line). This kind of suspension prevents rotatory motion of the bodies, which would complicate the measurement of the velocities. The test bodies are suspended with help of a mounting gauge. The threads are adjusted precisely with pegs which are normally used to tune stringed instruments. The balls are made of high-alloy steel (1.4034) bearing balls with a diameter of 50mm, a mass of 504.491g and a hardness of approximate 54–60HRC. Adapter plates made of aluminium with a mass of 1.949g per plate are used to suspend the balls in a reproducible way. For position measuring of the colliding bodies a 30mm long steel scale tape with a mass of 0.890g is glued to the ball, which is needed for an exposed linear encoder, cf. figure 4. This is the only part generating a negligible small asymmetry. The assembling of the balls with the plates and the steel scale tape is done in a custom-built mounting gadget. The accurate mounting reduces noise in the signal of the contact-free linear encoder to a minimum. This position measuring system consists of a steel scale tape and a sensor generating a sinusoidal signal by means of the Moiré effect. The signal is processed by a PCI–counting card IK220. One signal period equals a distance of 20µm. The accuracy of the exposed linear encoder (LIDA 48, Fa. Heidenhain) is stated by the manufacturer to be within 1% of the signal period, thus 0.2µm. The sampling frequency of the counting card is 10kHz which is equivalent to a time interval of 100µs. In single shot mode the IK220 stores 8192 position values which equals at the highest sampling rate a measuring duration of 0.8191s. The velocities of the bodies are obtained by discrete derivation differences of the positions sampled by the IK220. The drawback of this procedure is noise in the velocities which has to be taken care of in the following data processing.

To measure the contact times the test bodies are connected via thin wires to a test circuit. Each closed contact bypasses one resistor of the test circuit. The resistors are chosen such, that the total resistance of the system provides a binary encoding of the set of active contacts $H(t(q))$. Thus, the measured voltage on a sensor resistor $R_a$ enables a unique identification of the actual configuration (closed or open contacts).

A vacuum pump with a steel nozzle is used to detach the first body of the chain to be tested. Preliminary test showed, that an elastic vacuum cup does not guarantee a satisfactory let off of the test body. To start the
experiment the pump is switched off by an electromagnetic valve in the air supply tube. Thus the setup on the optical table is not disturbed by mechanical action.

3.2 Results

<table>
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<th>n</th>
<th>$\epsilon_{11}$</th>
<th>$\epsilon_{21}$</th>
<th>$\epsilon_{31}$</th>
<th>$\epsilon_{41}$</th>
<th>$\Delta t$ [µs]</th>
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<td>0.0250</td>
<td>0.1261</td>
<td>0.8672</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Measured matrix elements of the Frémond matrix $\epsilon$ and contact times for the $n$-ball chains.

In every case there was almost conservation of energy, $T_r^+ \approx T_r^-$. Most of the energy was transferred in the last ball, see table 1 and figure 5. For the 3-, 4- and 5-ball chain the ball before the last one moved slowly forwards. All other balls remained at rest or moved slowly backwards. All contacts opened during the impact. The contact times $\Delta t$ confirmed that deformation of the contacts (Hertz theory) is responsible for the dynamics and not (body) wave effects, see table 1.

Figure 5: Velocities for a collision in the first contact of a 3–ball chain.

ACKNOWLEDGMENT

This research was conducted within the European project SICONOS IST–2001–37172 and supported by the Swiss State Secretariat for Education and Research (SER).

References


