Master Thesis
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Stability Analysis of Thermoacoustic Coupling in Damper-Equipped Combustion Chambers Including Nonlinearities

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Preface

Modern gas turbines are prone to combustion instabilities, which yields significantly high amplitude pressure oscillations which reduce the structural lifetime. An efficient way to abate the instability is the use of Helmholtz dampers. These acoustic resonators are commonly used, yet it is a challenging task to design the dampers during the development phase of a new burner. Therefore, Nicolas Noiray and Bruno Schuermans [9] from ALSTOM Power have derived a model consisting of a combustion chamber coupled to a Helmholtz damper. The model is greatly simplified, yet capable of sufficiently capturing the main dynamics. Linear analysis is generally used in the first design step. Nevertheless, the nonlinearities present in the system might yield actual behaviour far from the one being predicted using linear description. Stability and bifurcation analysis is used in this project to understand and suppress thermoacoustic instabilities.

This thesis has been written at the Department of Mechanical Engineering of the ETH Zurich. At this point, I would like to express my utmost gratitude to Prof. Dr. ir. habil. R.I. Leine for his supervision, advice and guidance throughout the thesis. I have highly benefited from his inexhaustible patience and profound knowledge and I am deeply grateful for all the fruitful discussions. Thanks to him, I could consolidate and enhance my knowledge in the field of nonlinear dynamics and additionally I could improve in precise working and scientific writing. I also gratefully acknowledge Prof. Ch. Glocker for giving me the opportunity to write this thesis at the Institute for Mechanical Systems of the ETH Zurich.

Zurich, August 2012

Michael Baumann
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Abstract

Combustion chambers of gas turbines can be viewed as organ pipes in which acoustic pressure and velocity oscillations can be sustained. Thermoacoustic instabilities can occur due to thermoacoustic coupling between the flame and the pressure oscillations. Acoustic resonators such as Helmholtz dampers are commonly used to increase the acoustic dissipation and abate the severe pressure oscillations. Yet, it remains a challenging task to derive conditions which promote the potential coupling. In order to predict the damper performance, a theoretical model combining the combustion instability and the damper dynamics is derived by Nicolas Noiray and Bruno Schuermans [9] from ALSTOM Power.

The model consists of two nonlinear coupled differential equations of second order. The first equation describes the acoustic pressure oscillation in the combustor of one particular acoustic mode. The linear growth rate is captured by a linear damping term, whereas the boundedness of the solutions are ensured by a nonlinear damping term. The second equation describes the acoustic velocity oscillation in the neck of the Helmholtz damper, which includes a non-smooth damping term. The acoustic absorption is maximized when the eigenfrequency of the damper is adjusted to the unstable mode of the combustor. This special case is referred to as the perfectly tuned model, which is a special case of the general model including static detuning.

The objective of this project is the nonlinear analysis of these models. The stability of the equilibrium is determined by an eigenvalue analysis and the region of attraction is approximated with the use of Lyapunov techniques. Due to the nonlinearities present in the system, several periodic solutions exist, which are approximated using three different methods from nonlinear dynamics, which are the center manifold reduction, the method of multiple scales and the method of averaging.

The bifurcation analysis of the perfectly tuned model shows a wider range of dynamics than expected. Additionally to a supercritical Hopf bifurcation, also fold bifurcations and a cusp catastrophe can occur. The bifurcation branches define a partition of the parameter space into four regions of which every region has a different number of limit sets.

The model with static detuning shows a similar behaviour. The pressure in
the combustion chamber and the acoustic velocity in the neck of the Helmholtz damper oscillate at the same frequency. Yet, which is in contrast to the perfectly tuned model, there exists a phase shift between the oscillations and the frequency of oscillation depends on the bifurcation parameter.
Nomenclature

Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( I )</td>
<td>identity matrix</td>
</tr>
<tr>
<td>( p' )</td>
<td>acoustic pressure</td>
</tr>
<tr>
<td>( p )</td>
<td>right eigenvector</td>
</tr>
<tr>
<td>( q' )</td>
<td>volumetric heat release rate fluctuation</td>
</tr>
<tr>
<td>( q )</td>
<td>left eigenvector</td>
</tr>
<tr>
<td>( s )</td>
<td>Laplace variable</td>
</tr>
<tr>
<td>( T )</td>
<td>period time</td>
</tr>
<tr>
<td>( u' )</td>
<td>acoustic velocity</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>eigenvalue</td>
</tr>
<tr>
<td>( \nabla )</td>
<td>nabla operator</td>
</tr>
<tr>
<td>( \nabla^2 )</td>
<td>Laplace operator</td>
</tr>
<tr>
<td>( \forall )</td>
<td>for all</td>
</tr>
<tr>
<td>( \exists )</td>
<td>there exists</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>set of real numbers</td>
</tr>
</tbody>
</table>

Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( x )</td>
<td>scalar</td>
</tr>
<tr>
<td>( \mathbf{x} )</td>
<td>column vector in ( \mathbb{R}^n )</td>
</tr>
<tr>
<td>( \mathbf{X} )</td>
<td>matrix in ( \mathbb{R}^{m \times n} )</td>
</tr>
<tr>
<td>( f(\mathbf{x}) )</td>
<td>single-valued function ( \mathbb{R}^n \rightarrow \mathbb{R} )</td>
</tr>
<tr>
<td>( f(\mathbf{x}) )</td>
<td>single-valued function ( \mathbb{R}^n \rightarrow \mathbb{R}^m )</td>
</tr>
<tr>
<td>( \text{det}(\mathbf{A}) )</td>
<td>determinant of ( \mathbf{A} )</td>
</tr>
<tr>
<td>( x_i )</td>
<td>( i )th element of ( \mathbf{x} )</td>
</tr>
<tr>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( |\mathbf{x}| )</td>
<td>Euclidean norm of ( \mathbf{x} )</td>
</tr>
</tbody>
</table>
\( x^T \)         transpose of \( x \)
\( \bar{x} \)    mean value of \( x \) or complex conjugate of \( x \)
\( x' \)        fluctuating part of \( x \)
\( x^* \)      complex conjugate of \( x \) or equilibrium point
\( \hat{x} \)    Fourier transform of \( x \)
\( \mathcal{L}[] \)  Laplace transformation
\( \mathcal{O}(\cdot) \)  Landau order symbol (big O notation)
\( \Re(x) \)    real part of \( x \)
\( \Im(x) \)    imaginary part of \( x \)
:=    definition
\( \dot{s}(\cdot) := \frac{1}{T} \int_{<T>} (\cdot) \sin \omega t \, dt \), (averaging operator)
\( \dot{c}(\cdot) := \frac{1}{T} \int_{<T>} (\cdot) \cos \omega t \, dt \), (averaging operator)

**Acronyms and Abbreviations**

LAS    Locally Asymptotically Stable
LHS    Left-Hand Side
\( cc \)    complex conjugate
CM(R)  Center Manifold (Reduction)
FTF    Flame Transfer Function
GAS    Globally Asymptotically Stable
H.O.T  Higher Order Terms
\( nst \)    non-secular terms
(L)PD   (Locally) Positive Definite
(L)PDF  (Locally) Positive Definite Function
RHS    Right-Hand Side
Chapter 1

Introduction

Dealing with combustion instabilities is a challenging task in the development of systems using continuous combustion processes like gas turbines. This problem gains in importance due to progressively more stringent low emission and high-power requirements. To meet these requirements modern system use premixing of fuel and air prior to combustion and operate lean and near stoichiometric. Especially these novel systems are prone to combustion instabilities which are manifested by augmented heat release and pressure oscillations. The resulting structural vibrations, high level acoustic noise and increase in heat fluxes reduce the system lifetime and often lead to severe damage such as mechanical failures or component melting. These oscillations occur due to constructive interference between flames and acoustic eigenmodes of the combustor. This interaction between combustion and acoustics is referred to as thermoacoustic coupling.

1.1 Thermoacoustic Instabilities

The coupling mechanisms leading to thermoacoustic instabilities are explained in [7, 10]. A schematic representation of the coupling mechanism is shown in Figure 1.1. The combustion dynamics describes the flame response, i.e. the relation between fluctuations in the heat release rate $q'$ and velocity perturbations $u'$. For linear flame responses this dynamics is commonly captured by the flame transfer function FTF, which is defined as the normalized ratio of the fluctuations in the heat release rate and the acoustic velocity. Flames are essentially surfaces across which reactants are converted into products. The heat release can be perturbed by flame surface area fluctuations, which itself is sensitive to the acoustic velocity. The fluctuations in heat release rate induce unsteady gas expansions which generate acoustic pressure oscillations $p'$. This process is referred to as combustion noise. The pressure fluctuations propagate in the combustion chamber and get reflected and damped at the boundaries or
1.1. Thermoacoustic Instabilities

Figure 1.1: Schematic diagram of thermoacoustic coupling. The states are acoustic pressure $p'$, acoustic velocity $u'$ and fluctuating heat release rate $q'$.

at area discontinuities. The acoustic pressure and velocity oscillations can be sustained like in an organ-pipe and their coupling is described by the burner acoustics.

Investigating combustion oscillations dates back to 1802 when Higgins [2] reported the phenomenon of the ‘singing flame’. In an experimental setup a gas flame was placed in a tube open at both ends. The flame exited a harmonic oscillation in the tube and, therefore, producing a high level of sound. In the middle of the nineteenth century LeConte [4] observed the influence of audible pressure waves to the flame of a gas lamp. In 1859, Rijke [15] discovered that sound can be sustained in a cylindrical tube open at both ends (similar experimental setup as it was used by Higgins) by placing a hot metal gauze in the lower half of the vertical tube. Rayleigh [12] stated his famous criterion in 1878 indicating the importance of the phase between the pressure oscillation and the unsteady heat release in encouraging the instability. His exact wording was:

“If heat be periodically communicated to, and abstracted from, a mass of air vibrating (for example) in a cylinder bounded by a piston, the effect produced will depend upon the phase of vibration at which the transfer of heat takes place. If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged.”

The Rayleigh Criterion [13] can be described in mathematical form by the
1.2. Acoustic Dampers

The LHS of (1.1) captures the power transformed from the fluctuating heat release rate \( q' \) to the pressure oscillations \( p' \) during one oscillation cycle with period time \( T \) inside the control volume \( V \). While the original criterion does not consider any dissipation in the acoustic field, this effect is captured by the wave energy dissipation \( \Phi \) in the RHS of (1.1). Thus, when the acoustic pressure and the heat release rate oscillations are in phase, total mechanical energy is added to the oscillation. If the energy addition overcomes the total energy dissipation, the acoustic energy in the combustor is increased, which leads to thermoacoustic instability of the combustion process.

1.2 Acoustic Dampers

In 1955, Putnam and Dennis [11] wrote the following comment:

“Combustion systems often generate acoustical oscillations; these oscillations not only may be annoying, but at times may become so violent as to damage or destroy the equipment. It is difficult, if not impossible, to design a combustion system in such a way that no oscillations can occur.”

Even though the problem of combustion instabilities has been widely investigated, this comment still holds true for most continuous combustion processes.

There exist several passive and active means of control to abate the instability. An efficient way is the use of acoustic resonators (e.g. Helmholtz dampers), which are commonly used in gas turbine applications. The main idea of acoustic dampers is to increase the acoustic dissipation (i.e. increase the term \( \Phi \) in the Rayleigh Criterion (1.1)) and, therefore, to decrease the acoustic energy in the system.

The ability of acoustic resonators to stabilize the system has to be evaluated prior to their implementation. Yet, combustion instabilities are difficult to predict during the development phase of a new burner. Therefore, theoretical models of the combustion instabilities and the damper dynamics are needed in order to optimize the number, dimensions and locations of dampers.

Generally, linear analysis is used in the first acoustic design step. The eigenfrequency of the dampers are adjusted to the unstable mode, where the acoustic absorption is maximized, and the stability of the equilibrium is determined by an eigenvalue analysis. It is nevertheless important to consider the nonlinearities present in the system, since they significantly influence the system behaviour away from the equilibrium. They determine the amplitude
and, thus, the boundedness of oscillations. In this context, three different nonlinearities are considered, i.e. (1) nonlinear response of the flame to the acoustic perturbation (nonlinear FTF), (2) nonlinear damping behaviour of the acoustic resonator and (3) detuning of the damper due to hot gas inflow.

1.3 Problem Definition and Thesis Outline

The objective of this project is to investigate the stability of a model consisting of a combustion chamber coupled to a Helmholtz damper. The analysis includes the stability analysis of the equilibrium as well as the approximation of periodic solutions.

The thermoacoustic model is derived in Chapter 2 and it describes the acoustic pressure oscillation in the combustion chamber of one particular acoustic mode and the acoustic velocity oscillation in the neck of the Helmholtz damper. The decoupled pressure oscillation in the combustion chamber can be either stable or it can become unstable due to thermoacoustic instability. Yet, the oscillations stay bounded due to nonlinear damping of the van der Pol-type. The damping in the Helmholtz resonator consists of a linear and a nonlinear part, which is non-smooth. Three different dynamical models are considered. The most general system includes dynamical detuning, i.e. the eigenfrequency of the Helmholtz damper is time-dependent. The second system assumes a constant eigenfrequency of the Helmholtz damper which is generally different from the eigenfrequency of the combustion chamber. Therefore, it is a special case of the first system. The third system describes the perfectly tuned case, where we assume that the eigenfrequencies of both subsystems are identical.

The stability analysis is presented in Chapter 3. The stability of the equilibrium is analyzed using an eigenvalue analysis and the region of attraction is approximated using Lyapunov techniques.

For certain choices of system parameters, periodic solutions can occur. Three different methods are used to find the normal form of the bifurcations, to approximate the periodic solutions and to determine the stability of the oscillations. The first method is the center manifold reduction, which is presented in Chapter 4. The second method, which is the method of multiple scales, is shown in Chapter 5. The last method used is the method of averaging, also called the method of slowly changing phase and amplitude and it is described in Chapter 6.

Finally, a summary and some concluding remarks are given in Chapter 7.
Chapter 2

Derivation of the Model Equations

The thermoacoustic model investigated in this thesis consists of a combustion chamber coupled to a Helmholtz resonator. The derivation of the combustor dynamics is shown in Section 2.1 and the dynamics of the Helmholtz resonator is derived Section 2.2. The coupling of the two individual systems is presented in Section 2.3. Furthermore, some additional nonlinearities are introduced and the system is normalised such that only dimensionless variables and dimensionless parameters remain in the model equations, which is presented in Section 2.4.

2.1 Dynamics of the Combustor

The goal of this section is to find a simple model for the acoustic propagation in the combustor, which is achieved in four main steps. In the first step the forced acoustic wave equation is derived. The equations are transformed into the frequency domain in the second step and the boundary conditions are introduced. Making use of a Green’s function, the partial differential equation is solved in the third step. Making some relevant simplifications, the dynamics of the combustor can be simplified to an ordinary differential equation, which is presented in the last step. The derivation of the unforced acoustic wave equation can be found in [5], whereas steps 2-4 are taken from [9].

In the following, it is assumed that the viscous stresses and the thermal conductivity to the surroundings can be neglected. Under these assumptions, the gas dynamics can be described by the conservation equations for mass, momentum and energy, given by
2.1. Dynamics of the Combustor

\( \frac{Dp}{Dt} + \rho \nabla u = 0, \) \hspace{1cm} (2.1)

\( \rho \frac{Du}{Dt} + \nabla p = 0, \) \hspace{1cm} (2.2)

\( \rho \frac{De}{Dt} + p \nabla u = q, \) \hspace{1cm} (2.3)

where \( \rho, u, p \) and \( e \) are density, velocity, pressure and specific internal energy, respectively. The quantity \( q \) is the heat release rate per unit volume and the operator \( \frac{D}{Dt} \) is the material derivative defined as \( \frac{D}{Dt} = \frac{\partial}{\partial t} + u \nabla. \)

Assuming a perfect gas, we can use the gas state equation \( \frac{p}{\rho} = RT \) with \( R \) the gas constant and \( T \) the absolute temperature in order to write the specific internal energy as

\[ e = c_v T = \frac{c_v p}{\rho} = \frac{1}{\gamma - 1} \frac{p}{\rho}, \]

(2.4)

where \( c_v \) is the specific heat for constant volume and \( \gamma \) is the specific heat ratio. Substituting (2.4) into (2.3) and using (2.1), we have

\[ \frac{Dp}{Dt} + \gamma p \nabla u = (\gamma - 1) q. \] \hspace{1cm} (2.5)

We linearize (2.2) and (2.5) since we are only interested in fluctuating quantities. The mean parts of the variables are considered spatially constant and, furthermore, only small Mach number flows are considered. Hence, the mean velocity is neglected. The linearized equations (2.2) and (2.5) for the fluctuating quantities can be expressed as

\[ \rho \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = 0, \] \hspace{1cm} (2.6)

\[ \frac{\partial p'}{\partial t} + \gamma \rho \nabla \mathbf{u}' = (\gamma - 1) q', \] \hspace{1cm} (2.7)

where \( (\cdot)' \) and \( (\cdot) \) denote the fluctuating and the mean part of the corresponding variable, respectively. Introducing the average speed of sound \( \bar{c} = \sqrt{\gamma \frac{\bar{p}}{\bar{\rho}}} \) and evaluating \( \frac{D}{Dt} (2.7) - \bar{c}^2 \nabla (2.6), \) we obtain the forced acoustic wave equation as

\[ \left( \frac{\partial^2}{\partial t^2} - \bar{c}^2 \nabla^2 \right) p' = (\gamma - 1) \frac{\partial}{\partial t} q'. \] \hspace{1cm} (2.8)
2.1. Dynamics of the Combustor

Defining the Laplace transformed acoustic pressure \( \hat{p}(s, x) = \mathcal{L}[p'(t, x)] \), acoustic velocity \( \hat{u}(s, x) = \mathcal{L}[u'(t, x)] \) and fluctuating heat release rate \( \hat{q}(s, x) = \mathcal{L}[q'(t, x)] \), where \( s \) is the Laplace variable, we can rewrite (2.8) as

\[
\left( \nabla^2 - \left( \frac{s}{c} \right)^2 \right) \hat{p}(s, x) = -s^{\gamma - 1} \hat{q}(s, x), \tag{2.9}
\]

which is valid inside the combustor volume \( V \). The boundary condition on the surrounding surface \( S \) can be written as

\[
\hat{p}(s, x) \cdot \hat{u}(s, x) \cdot n = Z(s, x), \tag{2.10}
\]

where \( n \) is the outward normal on \( S \) and \( Z(s, x) \) is the acoustic impedance. The acoustic impedance is zero for an open end, since the fluctuating part of the pressure has to vanish. For a rigid wall, i.e. \( \hat{u} \cdot n = 0 \), the acoustic impedance tends to infinity.

We introduce the Green’s function \( \hat{G}(x|x_0) \) for the differential operator used in (2.9), which is defined as

\[
\left( \nabla^2 - \left( \frac{s}{c} \right)^2 \right) \hat{G}(x|x_0) = \delta(x - x_0), \tag{2.11}
\]

where \( \delta \) is the Dirac delta function. The corresponding boundary condition on \( S \) is

\[
\nabla \hat{G}(x|x_0) \cdot n = g(x). \tag{2.12}
\]

Here, the Green’s function can be interpreted as the impulse response of the linear system described by the forced wave equation. The left-hand side of the boundary condition (2.12) can be written as \( \nabla \hat{G} \cdot n = \frac{\partial \hat{G}}{\partial n} \), which is the directional derivative of \( \hat{G} \) in the direction of \( n \). The function \( g(x) \) will be chosen later according to the considered acoustic eigenmodes.

Evaluating \( \hat{G} \cdot (2.9) - \hat{p} \cdot (2.11) \), integrating over the volume \( V \) and using the symmetry property of the Green’s function \( \hat{G}(x_0|x) = \hat{G}(x|x_0) \), we have

\[
\hat{p}(s, x) = -s^{\gamma - 1} \int_V \hat{G}(x|x_0) \hat{q}(s, x_0) dV + \int_V \nabla \left( \hat{p}(s, x_0) \nabla \hat{G}(x|x_0) \right) dV - \int_V \nabla \left( \hat{G}(x|x_0) \nabla \hat{p}(s, x_0) \right) dV. \tag{2.13}
\]

Applying Stokes’ theorem, using the Laplace transform of (2.6) and substitut-
2.1. Dynamics of the Combustor

ing the boundary conditions (2.10) and (2.12) yields

\[ \hat{p}(s, \mathbf{x}) = -s \gamma - \frac{1}{c^2} \int_V \hat{G}(\mathbf{x}|\mathbf{x}_0) \hat{q}(s, \mathbf{x}_0) dV + \int_S \hat{p}(s, \mathbf{x}_0) g(\mathbf{x}_0) dS \]

\[ + s \bar{\rho} \int_S \hat{G}(\mathbf{x}|\mathbf{x}_0) \hat{p}(s, \mathbf{x}_0) \frac{1}{Z(s, \mathbf{x}_0)} dS. \]  

(2.14)

An orthogonal basis \( \Psi \) is introduced on which the Green’s function is spanned

\[ \hat{G}(\mathbf{x}|\mathbf{x}_0) = \sum_{j=0}^{\infty} A_j \Psi_j(\mathbf{x}), \]  

(2.15)

where the basis functions \( \Psi_j \) are defined as the solution to

\( \left( \nabla^2 + \left( \frac{\omega_j}{c} \right)^2 \right) \Psi_j = 0, \)  

(2.16)

together with the boundary condition

\[ \nabla \Psi_j \cdot \mathbf{n} = g. \]  

(2.17)

Note that the basis functions \( \Psi_j \) are the eigenfunctions to the eigenmodes with the corresponding characteristic frequency \( \omega_j \) without acoustic sources. Nevertheless, it is assumed that the Green’s function \( \hat{G} \) (and accordingly the acoustic pressure \( \hat{p} \)) can still be expressed in the orthogonal basis \( \Psi \) even if the acoustic source is present. This assumption is valid if the acoustic source is small.

Substituting (2.15) into (2.11) multiplied by \( \Psi_m^* \) and integrating over the volume \( V \) yields

\[ \int_V \left( \nabla^2 + \left( \frac{\omega_j}{c} \right)^2 \right) \sum_{j=0}^{\infty} (A_j \Psi_m^* \Psi_j) dV = \Psi_m^*(\mathbf{x}_0). \]  

(2.18)

The coefficients \( A_m \) are found by simplifying (2.18) using (2.16) and the orthogonality of the basis functions \( \Psi_j \). Therewith, we can write (2.15) as

\[ \hat{G}(\mathbf{x}|\mathbf{x}_0) = \sum_{j=0}^{\infty} -\frac{\Psi_j^*(\mathbf{x}_0) c^2}{V \Lambda_j (s^2 + \omega_j^2)} \Psi_j(\mathbf{x}), \]  

(2.19)
2.1. Dynamics of the Combustor

where

\[ \Lambda_j = \frac{1}{V} \int_V |\Psi_j|^2 dV. \tag{2.20} \]

Substituting (2.19) into (2.14) yields

\[
\hat{p}(s, x) = \sum_{j=0}^{\infty} \Psi_j(x) \left( \frac{s(\gamma - 1)}{s^2 + \omega_j^2} \frac{1}{V \Lambda_j} \int_V \hat{q} \Psi_j^* dV - s \frac{\bar{\rho} \bar{c}^2}{V \Lambda_j (s^2 + \omega_j^2)} \int_S \frac{1}{Z} \hat{p} \Psi_j^* dS \right) \\
+ \int_S \hat{p} g dS. \tag{2.21} \]

Since we consider only closed or open boundary conditions, the last term in (2.21) is always equal to zero. For the closed case, \( g \) has to vanish since the fluctuating velocity perpendicular to the wall tends to zero. For the open case, \( \Psi_k \) has to vanish, since the fluctuating pressure has to tend to zero at an open boundary. Omitting the last term in (2.21) and using the orthogonality of the basis functions \( \Psi_j \), we obtain the acoustic pressure in the orthogonal basis \( \Psi \) as

\[ \hat{p}(s, x) = \sum_{j=0}^{\infty} \eta_j(s) \Psi_j(x), \tag{2.22} \]

where

\[ \eta_j = \frac{s \bar{\rho} \bar{c}^2}{s^2 + \omega_j^2} \frac{1}{V \Lambda_j} \left( \frac{\gamma - 1}{\bar{\rho} \bar{c}^2} \int_V \hat{q} \Psi_j^* dV - \int_S \frac{1}{Z} \eta_j |\Psi_j|^2 dS \right). \tag{2.23} \]

In order to simplify (2.23), the following assumptions are made:

- The thermoacoustic coupling acts only on the eigenmode with the largest growth rate. All other eigenmodes are neglected and, thus, we have \( \eta_j = 0 \quad \forall j \neq k \).
- The fluctuating volumetric heat release rate \( \hat{q} \) can be decomposed as \( \hat{q} = \hat{q}_N + \hat{q}_C(\eta, \Psi) \), where \( \hat{q}_C(\eta, \Psi) \) is the coherent component correlated to the acoustic pressure and \( \hat{q}_N \) is the component independent of the acoustics. The component \( \hat{q}_N \) represents noise and is assumed to be negligible.
2.1. Dynamics of the Combustor

Using these assumptions and dropping the index \( j \) from now on, (2.23) becomes

\[
\eta = \frac{s\bar{\rho}c^2}{s^2 + \omega^2} \frac{1}{V\Lambda} \left( \frac{\gamma - 1}{\bar{\rho}c^2} \int_V \hat{q}_N(\eta, \Psi)\Psi^* dV - \int_{S - S_d} \frac{1}{Z} |\Psi|^2 dS \right) \tag{A} \\
- \int_{S_d} \frac{1}{Z} |\Psi|^2 dS \right) \tag{B} \\
- \int_{S_d} \frac{1}{Z} |\Psi|^2 dS \right) \tag{C}, \tag{2.24}
\]

where the boundary surface \( S \) is separated into a part \( S_d \), where the acoustic damper is placed, and the remaining part consisting of the surface \( S - S_d \).

Considering only one particular acoustic mode, the basis function \( \Psi \) has to be real and the pressure can be written as \( \hat{p}(s, x) = \eta(s)\Psi(x) \). Therefore, multiplying (2.24) by \( \eta \) and using equation (2.20), we obtain

\[
s \left( 1 + \left( \frac{\omega}{s} \right)^2 \right) \frac{1}{\bar{\rho}c^2} \int_V \hat{p}^2 dV = \frac{\gamma - 1}{\bar{\rho}c^2} \int_V \hat{q}_N(\eta, \Psi)\hat{p} dV - \int_{S - S_d} \frac{1}{Z} \hat{p}^2 dS \\
- \int_{S_d} \frac{1}{Z} \hat{p}^2 dS. \tag{2.25}
\]

In order to analyse equations (2.24) and (2.25) and motivating further simplifications, the energy balance in the frequency domain\(^1\) is presented at this point. Calculating the Laplace transform of (2.6) and (2.7) and evaluating \( \hat{u} \cdot \mathcal{L}[\hat{v}] + \frac{1}{\bar{\rho}c^2} \hat{p} \cdot \mathcal{L}[\hat{p}] \) together with \( \gamma\bar{\rho} = \bar{\rho}c^2 \) yields

\[
s \left( \frac{1}{\bar{\rho}c^2} \hat{\rho}^2 + \hat{\rho} \hat{u}^2 \right) = \frac{\gamma - 1}{\bar{\rho}c^2} \hat{q} - \nabla (\hat{p} \hat{u}). \tag{2.26}
\]

Calculating the Laplace transform of (2.6), multiply with \( \hat{p} \) and using (2.16) together with (2.22), we obtain

\[
- \frac{1}{s} \hat{u} \nabla^2 \hat{p} = \frac{1}{\bar{\rho}c^2} \left( \frac{\omega}{s} \right)^2 \hat{p} \nabla \hat{p}. \tag{2.27}
\]

Substituting again the Laplace transform of (2.6) into (2.27) and making use of the product rule, it can be shown that

\[
\hat{\rho} \hat{u}^2 = \frac{1}{\bar{\rho}c^2} \left( \frac{\omega}{s} \right)^2 \hat{p}^2. \tag{2.28}
\]

\(^1\)The Rayleigh criterion (1.1) presented in Section 1.1 can easily be derived from the energy balance in the time domain.
2.1. Dynamics of the Combustor

Integrating (2.26) over the control volume $V$, applying Stokes’ theorem and making use of (2.28), we have

$$s\left(1 + \left(\frac{\omega}{s}\right)^2\right)\frac{1}{\bar{\rho}c^2} \int_V \hat{p}^2 dV = \frac{\gamma - 1}{\bar{\rho}c^2} \int_V \hat{q}\hat{p} dV - \int_S \hat{p} (\hat{u} \cdot \hat{n}) dS.$$  \hspace{1cm} (2.29)

The LHS of (2.29) is the change in the acoustic energy, which is equal to the difference of the acoustic power transformed from the fluctuating heat release rate to the pressure oscillation (first integral of the RHS of (2.29)) and the acoustic energy flux across the control surface $S$ (second integral of the RHS of (2.29)).

We recall the definition of the acoustic impedance (2.10) $Z = \frac{\hat{p}}{\hat{u} \cdot \hat{n}}$ and we note that the equations (2.25) and (2.29) are identical. This motivates us to state the following additional assumption.

- The influence of the constructive/destructive interaction between the coherent component of the fluctuating heat release rate $\hat{q}_N$ and the acoustics (integral (A) in (2.24)) together with the acoustic energy flux across the control surface without the damper $S - S_d$ (integral (B) in (2.24)) can be captured by introducing a nonlinear damping term $\hat{f}(\eta, s\eta)$.

Using this assumption, (2.24) becomes

$$\eta = s\bar{\rho}c^2 \left(\frac{1}{s^2 + \hat{f}(\eta, s\eta)s + \omega^2} \right) \frac{1}{V\Lambda} \left( - \int_{S_d} \frac{1}{Z} |\Psi|^2 dS_d \right).$$  \hspace{1cm} (2.30)

The a Helmholtz damper with an acoustic impedance $Z_d(s)$ is placed at the position $\mathbf{x}_d$. The cross-section of the neck $S_d$ is assumed to be small compared to the considered wavelength and, thus, the eigenfunction $\Psi(\mathbf{x})$ is considered constant over $S_d$, which yields

$$\eta = -\frac{s\bar{\rho}c^2}{s^2 + \hat{f}(\eta, s\eta)s + \omega^2} \frac{S_d |\Psi(\mathbf{x}_d)|^2}{V\Lambda Z_d^2} \eta.$$  \hspace{1cm} (2.31)

We now introduce a non-dimensional parameter $\epsilon$ describing the damping efficiency

$$\epsilon = \frac{V_d |\Psi(\mathbf{x}_d)|^2}{V \Lambda}.$$  \hspace{1cm} (2.32)

Additionally, as we will see in Section 2.2 (equation (2.43)), the eigenfrequency of the damper is given by

$$\omega_d^2 = \tilde{c}_d^2 \frac{S_d}{V_d d_d}.$$  \hspace{1cm} (2.43)
2.2. Dynamics of the Helmholtz Resonator

Noting that \( \bar{\rho}c^2 \approx \bar{\rho}_d \bar{c}_d^2 \) and making use of (2.32) and (2.43), equation (2.31) can be written as

\[ \eta = -\frac{\epsilon \omega_d \bar{\rho}_d l_d}{s^2 + f(\eta, s\eta)s + \omega^2 Z_d} \frac{s\eta}{Z_d}. \]  \hspace{1cm} (2.33)

Defining the acoustic velocity \( \hat{u}_d'(s) \) at the damper location in the direction of the outward normal \( n \), i.e. \( \hat{u}_d'(s) = \hat{u}(s, x_d) \cdot n \), we can write the acoustic impedance of the damper \( Z_d \) as

\[ Z_d(s, x_d) = \frac{\hat{p}(s, x_d)}{\hat{u}(s, x_d) \cdot n} = \frac{\eta(s)\Psi(x_d)}{\hat{u}_d'(s)}. \]  \hspace{1cm} (2.34)

Substituting (2.34) into (2.33) and applying the inverse Laplace transform yields the dynamics of \( \eta(t) \) in the time domain as

\[ \frac{d^2}{dt^2} \eta + f(\eta, \frac{d}{dt}\eta) \frac{d}{dt}\eta + \omega^2 \eta = -\epsilon \omega_d^2 \frac{\bar{\rho}_d l_d}{\Psi(x_d)} \frac{d}{dt}\eta_d. \]  \hspace{1cm} (2.35)

2.2 Dynamics of the Helmholtz Resonator

The Helmholtz resonator is a widely used acoustic resonator capable of enhancing absorption of pressure oscillations. Its shape is similar to the shape of a bottle. The gas in the body acts as an acoustic spring whereas the gas in the neck acts as an acoustic mass\(^2\). The damping of the resonator can be increased if a continuous volume flow is injected into the bottle. The main energy reservoirs are the pressure inside the bottle and the inertia of the gas volume inside the neck. Therefore, the dynamics of the Helmholtz resonator can be described by a second order differential equation, which is derived in this section. The derivation of the linear model can be found in [14].

We consider a Helmholtz resonator as depicted in Figure 2.1. It consists of a cavity with volume \( V_d \) and a small neck of length \( l_d \). Since the cross-sectional area of the bottle \( S_b \) is much larger than the cross-sectional area of the neck \( S_d \), the pressure perturbation inside the bottle \( \hat{p}_u \) can be assumed to be uniform. The compressibility inside the neck can be neglected because the length of the neck is short compared to the considered wave lengths. A continuous volume flow \( \bar{Q} = \bar{u} S_d \) is injected into the bottle. Furthermore, we consider the flow to be uniform and frictionless. When we apply the Bernoulli equation for unsteady potential flow between a point at the entrance of the neck and a point at the exit of the neck, we have\(^3\) for

\(^2\)This interpretation is only valid if the dimensions of the resonator are small compared to the considered acoustic wave lengths.

\(^3\)Due to the inertia of the acoustic flow, a small end correction \( \delta \) is added to both ends.
2.2. Dynamics of the Helmholtz Resonator

Figure 2.1: Helmholtz resonator with mean flow. An uniform jet is assumed in the downstream of the neck.

\[ \bar{u} + u'_n > 0 : \bar{\rho}_d l_d \frac{d}{dt} u'_n + \frac{1}{2} \bar{\rho}_d (\bar{u} + u'_n)^2 + p'_{ex} = \bar{p} + p'_{in}. \quad (2.36) \]

The pressure inside the quasi-stationary jet is assumed to be constant and equal to the external pressure, which is reasonable for \( \bar{u}/c << 1 \). If the fluctuation \( u'_n \) becomes smaller than \( -\bar{u} \), the jet is formed inside the resonator and the Bernoulli equation yields for

\[ \bar{u} + u'_n < 0 : \bar{\rho}_d l_d \frac{d}{dt} u'_n + \bar{p} + p'_{ex} = \frac{1}{2} \bar{\rho}_d (\bar{u} + u'_n)^2 + p'_{in}. \quad (2.37) \]

Gathering the zero order terms of the acoustic perturbation in (2.36) and (2.37), we have

\[ \bar{p} = \frac{1}{2} \bar{\rho}_d \bar{u}^2 \quad (2.38) \]

and for the first order terms, we have

\[ \bar{\rho}_d l_d \frac{d}{dt} u'_n + \bar{\rho}_d (\bar{u}u'_n + \frac{1}{2} u^2_n) \text{sgn}(\bar{u} + u'_n) + p'_{ex} = p'_{in}. \quad (2.39) \]

The linearized mass conservation law for the volume \( V_d \) can be written as

\[ V_d \frac{d\rho'_\text{in}}{dt} = -\bar{\rho}_d u'_n S_d. \quad (2.40) \]

Since the compression of the gas inside the bottle is assumed to be adiabatic, the constitutive equation holds; that is

\[ p'_{\text{in}} = c_d p'_{\text{in}}. \quad (2.41) \]

We introduce a tuning parameter \( \zeta \), which can be interpreted as a damping efficiency. Deriving (2.39) with respect to time and substituting (2.40) and of the neck as \( l_d := l_d + 2\delta \).
2.3 Coupling of the Combustor with the Helmholtz Damper including Detuning

(2.41), we obtain the dynamics of the Helmholtz resonator as

\[
\frac{d^2}{dt^2} u_n' + \frac{1}{l_d} [\bar{u} + u_n'] \frac{d}{dt} u_n' + \omega_d^2 u_n' = -\frac{1}{\bar{\rho} d_d} \frac{d}{dt} p_{ex}',
\]

(2.42)

where the resonance frequency \( \omega_d \) is given as

\[
\omega_d^2 = \frac{c_d^2 S_d}{V_d l_d}.
\]

(2.43)

2.3 Coupling of the Combustor with the Helmholtz Damper including Detuning

We consider a system consisting of a combustion chamber coupled with a Helmholtz damper as depicted in Figure 2.2. In order to find the model equations of the overall system, we first recall the dynamics of the combustor

\[
\frac{d^2}{dt^2} \eta + f \left( \eta, \frac{d}{dt} \eta \right) \frac{d}{dt} \eta + \omega^2 \eta = -\epsilon \omega_d^2 \frac{\bar{\rho} d_d}{\Psi(x_d)} \frac{d}{dt} u_d',
\]

(2.35)

and the dynamics of the Helmholtz damper

\[
\frac{d^2}{dt^2} u_n' + \frac{1}{l_d} [\bar{u} + u_n'] \frac{d}{dt} u_n' + \omega_d^2 u_n' = -\frac{1}{\bar{\rho} d_d} \frac{d}{dt} p_{ex}'.
\]

(2.42)

The two individual systems are coupled by the acoustic velocity \( u_n' \) in the neck of the Helmholtz damper and the acoustic pressure \( p'(x_d) = \eta \Psi(x_d) \) in the combustion chamber at the location of the resonator \( x_d \). The acoustic velocity \( u_d' \) in (2.35) is equal to \(-u_n'\), since they were introduced in opposite directions. The acoustic pressure \( p_{ex}' \) in (2.42) is the pressure outside the neck of the Helmholtz damper and, therefore, it can be expressed as \( p_{ex}' = \eta \Psi(x_d) \).

According to [8], the nonlinear damping term \( f(\eta, \frac{d}{dt} \eta) \) can be expanded in a Taylor series. When applying averaging to the unforced equation for the pressure oscillations, i.e. \( \frac{d^2}{dt^2} \eta + f(\eta, \frac{d}{dt} \eta) \frac{d}{dt} \eta + \omega^2 \eta = 0 \), the first nonlinear terms of \( f(\eta, \frac{d}{dt} \eta) \) appearing in the amplitude equation are \( \eta^2 \) and \( (\frac{d}{dt} \eta)^2 \).

Thus, we define the nonlinear damping term as

\[
f(\eta, \frac{d}{dt} \eta) = \kappa_0 + \kappa_1 \eta^2 + \kappa_2 \left( \frac{d}{dt} \eta \right)^2,
\]

(2.44)
2.3. Coupling of the Combustor with the Helmholtz Damper including Detuning

Figure 2.2: Combustion chamber (left) with a fluctuating heat release rate $q'$ and an acoustic pressure $p'$ coupled to a Helmholtz damper (right) with an acoustic velocity $u'_d$.

where $\kappa_0$, $\kappa_1$ and $\kappa_2$ are unknown constants. Thus, we can write the model equations of the coupled system as

$$
\frac{d^2}{dt^2} \eta + \left( \kappa_0 + \kappa_1 \eta^2 + \kappa_2 \left( \frac{d}{dt} \eta \right)^2 \right) \frac{d}{dt} \eta + \omega^2 \eta = -\epsilon \omega_d^2 \frac{\bar{\rho}_d l_d}{\Psi(x_d)} \frac{d}{dt} u'_d, \quad (2.45)
$$

$$
\frac{d^2}{dt^2} u'_d + \frac{\zeta}{l_d} |\bar{u} + u'_n| \frac{d}{dt} u'_d + \omega^2 u'_d = -\omega_d^2 \frac{\bar{\rho}_d l_d}{\bar{\rho}_d l_d} \frac{d}{dt} \eta, \quad (2.46)
$$

Note that by replacing $u'_n$ by $-u'_d$, the direction of positive mean flow $\bar{u}$ changes its sign. Therefore, the damping term $|\bar{u} + u'_n|$ in (2.42) becomes $|\bar{u} + u'_d|$ in (2.46).

The temperature inside the combustion chamber is generally higher than the temperature inside the Helmholtz damper. The continuous volume flow injected into the Helmholtz damper prevents the hot gas from reaching into the damper volume. However, for high acoustic amplitudes this is not the case anymore. When the temperature inside the damper rises, the density is lowered and, hence, the eigenfrequency of the resonator is increased. This can easily be seen using the spring pendulum analogon mentioned in Section 2.2. When the density inside the resonator volume is decreased, the spring constant of the acoustic spring is increased. If the acoustic mass remains unchanged, the eigenfrequency increases. The eigenfrequency $\omega_d$ was calculated as

$$
\omega_d^2 = \frac{\bar{c}_d^2}{\bar{V}_d l_d} S_d. \quad (2.43)
$$

We define the unperturbed eigenfrequency $\omega_{d,0}$ as

$$
\omega_{d,0}^2 = \frac{\bar{c}_{d,0}^2}{\bar{V}_{d,0} l_d} S_d. \quad (2.47)
$$

Using the definition of the speed of sound, i.e. $\bar{c} = \sqrt{\gamma RT}$, we can write (2.43)
2.3. Coupling of the Combustor with the Helmholtz Damper including Detuning

Figure 2.3: Approximate dependence of the damper eigenfrequency disturbance $\delta \omega_d(U'_d)$ on the integrated acoustic velocity $U'_d$. The disturbance can be written as $\delta \omega_d = \frac{\omega^2_d - \omega^2_{d,0}}{\omega^2_{d,0}}$.

with (2.47) as

$$\omega^2_d = \omega^2_{d,0} \frac{T_d}{T_{d,0}} = \omega^2_{d,0} \left(1 + \frac{T_d - T_{d,0}}{T_{d,0} \delta \omega_d}\right), \quad (2.48)$$

where $T_d$ and $T_{d,0}$ denote the perturbed and unperturbed temperature of the resonator, respectively.

In order that gas is ingested, the net acoustic flow into the damper during one pressure cycle (with approximately the eigenfrequency $\omega$) has to be greater than the mean flow $\bar{u}$. On the other hand, the temperature has to saturate when it reaches the temperature of the ingested hot gas. This motivates us to approximate the disturbance $\delta \omega_d$ in (2.48) as it is shown in Figure 2.3, where $U'_d$ is defined as

$$U'_d := \int u'_d dt, \quad (2.49)$$

$T_H$ is the temperature of the ingested hot gas and $D_d$ is the depth of the Helmholtz resonator. Substituting (2.48) into (2.45)-(2.46) and expanding the system with (2.49), we obtain the model equations of the coupled system.
2.4 Normalisation of the Coupled Dynamics

including detuning as

\[
\frac{d^2}{dt^2} \eta + \left( \kappa_0 + \kappa_1 \eta^2 + \kappa_2 \left( \frac{d}{dt} \eta \right)^2 \right) \frac{d}{dt} \eta + \omega^2 \eta = -\epsilon \omega_d^2(U'_d) \bar{\rho}_d d \frac{d}{dt} u'_d, \tag{2.50}
\]

\[
\frac{d^2}{dt^2} u'_d + \frac{\zeta}{l_d} \bar{u} + u'_d \frac{d}{dt} u'_d + \omega_d^2(U'_d) u'_d = \frac{\Psi(x_d)}{\bar{\rho}_d d} \frac{d}{dt} \eta, \tag{2.51}
\]

\[
\frac{d}{dt} U'_d = u'_d, \tag{2.52}
\]

where

\[
\omega_d^2(U'_d) = \omega_{d,0}^2 \left( 1 + \delta_{\omega_d}(U'_d) \right). \tag{2.53}
\]

Note that the dynamics (2.50)-(2.53) has infinitely many equilibria. One possibility to eliminate this behaviour and obtain a single equilibrium is to define \( U'_d \) as the output of a low-pass filter applied to \( u'_d \) instead of the integral of \( u'_d \), that is

\[
\frac{d}{dt} U'_d = -\frac{1}{\tau} U'_d + u'_d, \tag{2.54}
\]

where the time constant \( \tau \) is a tuning parameter and can be determined by the steady state behaviour of the filter.

2.4 Normalisation of the Coupled Dynamics

The model equations of the coupled system (2.50)-(2.53) are cumbersome to analyse. Therefore, the time as well as the states and parameters are normalized in order to write the model equations in a more manageable way. The time is normalised by the inverse natural frequency of the combustor as \( \tau = \omega_t \).

Therefore, the time derivative becomes \( \frac{d}{dt} = \omega \frac{d}{d\tau} \). We change from Leibniz’s notation to Newton’s notation for differentiation, where we define \( \dot{\cdot} \) as the derivative w.r.t. the dimensionless time \( \tau \). Thus, (2.50)-(2.53) can be expressed as

\[
\ddot{\eta} + \left( \frac{\kappa_0}{\omega} + \frac{\kappa_1}{\omega} \eta^2 + \omega \kappa_2 \eta^2 \right) \dot{\eta} + \eta = -\epsilon \left( \frac{\omega_d}{\omega} \right)^2 \left( 1 + \delta_{\omega_d}(U'_d) \right) \bar{\rho}_d d \omega \frac{d}{d\tau} \bar{u}' \dot{\eta}, \tag{2.55}
\]

\[
\ddot{u}'_d + \frac{\zeta}{l_d \omega} \bar{u} + u'_d \dot{u}'_d + \left( \frac{\omega_d}{\omega} \right)^2 \left( 1 + \delta_{\omega_d}(U'_d) \right) u'_d = \frac{\Psi(x_d)}{\bar{\rho}_d d \omega} \dot{\eta}, \tag{2.56}
\]

\[
\dot{U}'_d = \frac{1}{\omega} u'_d. \tag{2.57}
\]
2.4. Normalisation of the Coupled Dynamics

We introduce a dimensionless acoustic velocity $\xi$ and its integral $\chi$ as

$$\xi = \bar{\rho} d \bar{x} d \omega \Psi(x_d) u_d'$$

and

$$\chi = \bar{\rho} d \bar{x} d \omega^2 \Psi(x_d) U_d'. \tag{2.59}$$

Furthermore, we define the parameters

$$q = \frac{\kappa_0}{\omega}, \tag{2.60}$$
$$c_1 = \frac{\kappa_1}{\omega}, \tag{2.61}$$
$$c_2 = \omega \kappa_2, \tag{2.62}$$
$$d = \frac{\zeta \bar{u}}{l_d \omega}, \tag{2.63}$$
$$\delta = \frac{\zeta \Psi(x_d)}{\bar{\rho} d l_d^2 \omega^2}. \tag{2.64}$$

Additionally, we define the frequency ratio $\theta(\chi)$ as

$$\theta(\chi) = \theta_0 (1 + \delta \theta(\chi)), \tag{2.65}$$

where

$$\theta_0 = \left(\frac{\omega_d}{\omega}\right)^2, \tag{2.66}$$
$$\delta \theta(\chi) = \delta \omega_d (U_d'). \tag{2.67}$$

Then, (2.55)-(2.57) can be expressed in a much simpler form; that is

**System I:** Model with detuning

$$\dot{\eta} + (q + c_1 \eta^2 + c_2 \eta^2) \dot{\eta} + \eta = -\epsilon \theta(\chi) \dot{\xi}, \tag{2.68}$$
$$\dot{\xi} + |d + \delta \xi| \dot{\xi} + \theta(\chi) \xi = \dot{\eta}, \tag{2.69}$$
$$\dot{\chi} = \xi. \tag{2.70}$$

Additionally, we define two special cases, where (a) only linear and (b) only nonlinear damping of the Helmholtz resonator is considered. Equations (2.68)
2.4. Normalisation of the Coupled Dynamics

and (2.70) remain unchanged and equation (2.69) becomes

**System I.a:** \( \ddot{\xi} + d \dot{\xi} + \theta(\chi)\xi = \dot{\eta}, \)  
\( (2.71) \)

**System I.b:** \( \ddot{\xi} + \delta |\xi| \dot{\xi} + \theta(\chi)\xi = \dot{\eta}, \)  
\( (2.72) \)

where we used that the parameter \( \delta \) is always positive.

When we are not interested in the detuning of the Helmholtz damper, we can neglect the term \( \delta \theta(\chi) \) and we have \( \theta(\chi) = \theta_0 \). Therefore, the additional state \( \chi \) can be dropped and we obtain a special case of System I; that is,

**System II:** Model without detuning

\[ \ddot{\xi} + (q + c_1 \eta^2 + c_2 \dot{\eta}^2) \dot{\eta} + \eta = -\epsilon \theta_0 \dot{\xi}, \]
\( (2.73) \)

\[ \ddot{\xi} + |d + \delta \xi| \dot{\xi} + \theta_0 \xi = \dot{\eta}, \]
\( (2.74) \)

and the special cases

**System II.a:** \( \ddot{\xi} + d \dot{\xi} + \theta_0 \xi = \dot{\eta}, \)  
\( (2.75) \)

**System II.b:** \( \ddot{\xi} + \delta |\xi| \dot{\xi} + \theta_0 \xi = \dot{\eta}. \)  
\( (2.76) \)

The acoustic absorption is maximized when the eigenfrequency of the damper is adjusted to the unstable mode of the pressure oscillation inside the combustion chamber, i.e. \( \theta(\chi) = 1 \). This is the ideal case and we will refer to it as

**System III:** Perfectly tuned model

\[ \ddot{\xi} + (q + c_1 \eta^2 + c_2 \dot{\eta}^2) \dot{\eta} + \eta = -\epsilon \dot{\xi}, \]
\( (2.77) \)

\[ \ddot{\xi} + |d + \delta \xi| \dot{\xi} + \xi = \dot{\eta}, \]
\( (2.78) \)

and

**System III.a:** \( \ddot{\xi} + d \dot{\xi} + \xi = \dot{\eta}, \)  
\( (2.79) \)

**System III.b:** \( \ddot{\xi} + \delta |\xi| \dot{\xi} + \xi = \dot{\eta}. \)  
\( (2.80) \)

The range of interest for the parameters (2.60)-(2.64) and (2.66) is listed in Table 2.1. According to Figure 2.3, \( \delta \theta(\chi) \) has the properties

\[ \delta \theta(0) = 0, \]
\( (2.81) \)

\[ \frac{d}{d\chi} \delta \theta(\chi) \geq 0. \]
\( (2.82) \)
2.4. Normalisation of the Coupled Dynamics

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<th>Maximum</th>
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<td>$c_2$</td>
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<tr>
<td>$d$</td>
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<td>0.15</td>
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<tr>
<td>$\delta$</td>
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<tr>
<td>$\epsilon$</td>
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<td>0.25</td>
</tr>
</tbody>
</table>

Table 2.1: Range of interest for the system parameters.

Therefore, according to (2.65), $\theta(\chi)$ has the properties

\[
\theta(0) = \theta_0, \quad (2.83)
\]

\[
\frac{d}{d\chi} \theta(\chi) \geq 0. \quad (2.84)
\]
Chapter 3

Stability Analysis

The first step in investigating the systems derived in Chapter 2 is to determine the stability of the equilibrium point, which is presented in the Section 3.1. This is achieved by linearizing the system around the equilibrium point and calculating the eigenvalues of the system matrix. The equilibrium point is asymptotically stable if all eigenvalues have negative real part. The results of the eigenvalue analysis are only valid in the neighborhood of the equilibrium, which can be arbitrarily small. In order to find the attractivity properties and an estimate of the region of attraction, the nonlinearities have to be taken into account. This analysis is conducted using Lyapunov techniques, which is presented in Section 3.2.

We characterize the stability of the equilibrium in the sense of Lyapunov (see [3]). An equilibrium is stable if all solutions starting nearby stay nearby and it is unstable otherwise. Moreover, it is said to be asymptotically stable if it is stable and, additionally, the solutions tend to the equilibrium as the time approaches infinity. More precisely we define the stability with the following \( \epsilon - \delta \)-argumentation.

**Definition 3.0.1 (Lyapunov stability)** The equilibrium point \( x = 0 \) of \( \dot{x} = f(x) \) is

- **stable if** \( \forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \) such that
  \[ \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0. \]

- **unstable if not stable.**

- **locally asymptotically stable if** it is stable and \( \delta \) can be chosen such that
  \[ \|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0. \]
3.1. Eigenvalue Analysis

- globally asymptotically stable if it is stable and
  \[ \lim_{t \to \infty} x(t) = 0 \quad \forall x(0) \in \mathbb{R}^n. \]

3.1 Eigenvalue Analysis

The stability of the equilibrium point is determined using Lyapunov’s indirect method. It is a simple procedure for determining the stability using the Jacobian matrix.

**Theorem 3.1.1** (Lyapunov’s first (indirect) method) Let \( x = 0 \) be an equilibrium point for \( \dot{x} = f(x) \), where \( f : D \to \mathbb{R}^n \) is continuously differentiable and \( D \) is a neighborhood of the origin. Let

\[
A = \frac{\partial f}{\partial x}(x) \bigg|_{x=0}.
\]

Then,

- the origin is locally asymptotically stable if \( \Re(\lambda_i) < 0 \) for all eigenvalues \( \lambda_i \) of \( A \).
- the origin is unstable if \( \Re(\lambda_i) > 0 \) for one or more eigenvalues of \( A \).

The proof of Theorem 3.1.1 can be found in [3].

In order to find the Jacobian matrix of the System II, we introduce a new state vector \( x \in \mathbb{R}^4 \) as

\[
x = (x_1 \ x_2 \ x_3 \ x_4)^T := (\eta \ \dot{\eta} \ \xi \ \dot{\xi})^T.
\]

(3.1)

Rewriting the equations (2.73) and (2.74) of System II as a first order differential equation in \( x \) yields

\[
\dot{x} = f(x) = \begin{pmatrix} \frac{x_2}{x_4} \\ -x_1 - (q + c_1 x_1^2 + c_2 x_2^2) x_2 - \epsilon \theta x_4 \\ x_2 - \theta x_3 - |d + \delta x_3| x_4 \end{pmatrix},
\]

(3.2)

where the index of \( \theta_0 \) is dropped from now onwards. Equation (3.2) is linearized around the equilibrium, which is located at the origin \( x^* = 0 \), thus we have

\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -q & 0 \\ 0 & 0 & 0 & -\epsilon \theta \\ 0 & 1 & -\theta & -d \end{pmatrix} x = A x.
\]

(3.3)
3.1. Eigenvalue Analysis

According to Theorem 3.1.1, the equilibrium of (3.2) is stable if all eigenvalues have negative real part. The eigenvalues $\lambda$ are the roots of the corresponding characteristic polynomial, which is obtained as

$$p(\lambda) = \det(\lambda I - A) = \lambda^4 + \left(\frac{a_1}{d + q}\right) \lambda^3 + \left(\frac{a_2}{\theta + q \theta + \epsilon \theta + 1}\right) \lambda^2 + \left(\frac{a_3}{d + q \theta}\right) \lambda + \left(\frac{a_4}{\theta}\right).$$

(3.4)

The Hurwitz criterion gives us necessary and sufficient conditions for which all roots of (3.4) have negative real part (see [1]), which are

1. $0 < a_1 = d + q$, (3.5a)
2. $0 < a_4 = \theta$, (3.5b)
3. $0 < a_1 a_2 - a_3 = (d + q)(\theta + q d + \epsilon \theta + 1) - (d + q \theta)$, (3.5c)
4. $0 < a_3 (a_1 a_2 - a_3) - a_1^2 a_4 = (d + q \theta)((d + q)(\theta + q d + \epsilon \theta + 1) - (d + q \theta)) - (d + q)^2 \theta$. (3.5d)

The conditions (3.5b) and (3.5d) imply that

$$a_3 (a_1 a_2 - a_3) > a_1^2 a_4 > 0$$

and, thus, we can rewrite condition (3.5c) as

$$a_3 > 0. \quad (3.5c^*)$$

Due to the given parameter range (see Table 2.1), the condition (3.5b) is always fulfilled. By solving (3.5d) for $\epsilon$, the criteria for stability can be written as

$$\mu := \epsilon \theta + dq \left(1 + \frac{(1 - \theta)^2}{(d + q)(d + q \theta)}\right) > 0, \quad (3.6)$$
$$\nu_1 := d + q > 0, \quad (3.7)$$
$$\nu_2 := d + q \theta > 0. \quad (3.8)$$

Note that for $q > 0$, the conditions (3.6)-(3.8) are satisfied for the considered parameter space. The contour lines $\nu_1 = 0$, $\nu_2 = 0$ and $\mu = 0$ in the $d$-$q$-plane are depicted in Figure 3.1 for different values of $\epsilon$. The contour line $\mu = 0$ for $\epsilon = 0$ coincide with the $d$- and $q$-axis. For an increasing value of $\epsilon$, they approach the contour lines $\nu_1 = 0$ and $\nu_2 = 0$. For the limit of $\epsilon$ tending to infinity, the contour line $\mu = 0$ coincide with the contour lines $\nu_1 = 0$ and $\nu_2 = 0$. 

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3.1. Eigenvalue Analysis

For System III, i.e. $\theta = 1$, the conditions (3.6), (3.7) and (3.8) reduce to

\[
\mu = \epsilon + dq > 0, \quad (3.9)
\]
\[
\nu = \nu_1 = \nu_2 = d + q > 0. \quad (3.10)
\]

Condition 3.9 gives a lower bound for the coupling efficiency $\epsilon$. If the dimensions of the Helmholtz resonator are too small compared to the combustion chamber, the parameter $\epsilon$ is small and the equilibrium becomes unstable. Interestingly, given that (3.9) is violated, increasing the linear damping $d$ of the Helmholtz resonator does not help. Condition 3.10 is fulfilled, if the sum of the two linear damping terms $d$ and $q$ is large enough. A high linear growth rate of the thermoacoustic instability demands a high linear damping of the Helmholtz resonator.

The contour lines $\nu = 0$ and $\mu = 0$ in the $d$-$q$-plane for the perfectly tuned case are depicted in Figure 3.2. For $\epsilon = 0$, the condition $\mu > 0$ can not be fulfilled for a negative $q$, whereas for $\epsilon > 0.0045$ the condition $\mu > 0$ is fulfilled for all considered values of $q$ and $d$ (see Table 2.1).

![Figure 3.1: Contour lines $\nu_1 = 0$, $\nu_2 = 0$ and $\mu = 0$ for different values of $\epsilon$ in the $q$-$d$-plane for System II with $\theta = 0.95$.](image)

The four eigenvalues of the system matrix $A$ for System II and System III for the considered values of $q$ appear as two complex conjugated pairs. Thus,
3.1. Eigenvalue Analysis

Figure 3.2: Contour lines \( \nu = 0 \) and \( \mu = 0 \) for different values of \( \epsilon \) in the \( q-d \)-plane for System III.
3.1. Eigenvalue Analysis

using the conditions (3.6)-(3.7), the parameter space can be divided into three regions:

**Region I:** \( \mu > 0 \land \nu_1 > \land \nu_2 > 0 \)
Both pairs of eigenvalues have negative real part.

**Region II:** \( \mu < 0 \)
One pair of eigenvalues has negative real part, while the other pair has positive real part.

**Region III:** \( \mu > 0 \land \nu_1 < 0 \land \nu_2 < 0 \)
Both pairs of eigenvalues have positive real part.

Figure 3.3 depicts these three regions in the \( q-d \)-plane for System II with \( \theta = 0.95 \) and \( \epsilon = 0.0025 \). The equilibrium is stable if the parameters are chosen in Region I. The borders between Region I and Region II as well as between Region II and Region III are given by \( \mu = 0 \). For \( \epsilon = 0 \), the entire depicted area belongs to Region II, whereas for the limit \( \epsilon \to \infty \) the Region II contracts to a cone defined by \( \nu_1 = 0 \) and \( \nu_2 = 0 \). The three regions for System III for \( \epsilon = 0.0025 \) are depicted in Figure 3.4. For this degenerate case the regions are similar as for the System II, but the Regions I and III have one common border defined by \( \nu = 0 \). If the parameters are chosen exactly on this common border, the characteristic polynomial becomes \( p(\lambda) = \lambda^4 + (2 + \mu)\lambda^2 + 1 = 0 \), which has only purely imaginary roots for \( \mu > 0 \). Therefore, all four eigenvalues are purely imaginary for \( \nu = 0 \).

Therefore, when the parameters are chosen inside Region I and the parameter \( q \) is continuously decreased, the equilibrium will eventually become unstable. For System II the change in stability occurs, because condition (3.6) is no longer fulfilled, while conditions (3.7) and (3.8) still hold true. For System III it depends on the parameters \( d \) and \( \epsilon \) which condition will be violated. For \( d^2 > \epsilon \), the condition (3.9) will be violated first and for \( d^2 < \epsilon \) the condition (3.10) is more restrictive.

The evolution of the eigenvalues under the influence of the parameter \( q \), i.e. moving along the dashed line in Figure 3.3 and 3.4, is shown in Figure 3.5 and Figure 3.6, respectively. The parameters are chosen as \( \epsilon = 0.0025 \) and \( d = 0.1 \) and the ratio of the eigenfrequencies for System II is chosen as \( \theta = 0.95 \). The arrows in the figures point in the direction of a decreasing value of \( q \). Only the two eigenvalues with positive imaginary part are shown, since the eigenvalues appear as complex conjugated pairs. For \( q = 0.03 \) (point 1), all eigenvalues are in the open left half complex plane for both cases, i.e. the parameters are inside Region I. At the point 2, i.e. \( q = 0 \), the two pairs of eigenvalues of the perfectly tuned model coincide, whereas for the System II both pairs of eigenvalues stay distinct. At \( q = -0.03 \) (point 4), one pair of eigenvalues has positive real part after entering Region II at point 3 (\( \mu = 0 \)). The eigenvalues crossing the imaginary axis are located at \( \pm i \sqrt{\frac{d+qd}{d+q}} \) for System II and at \( \pm i \) for System III.
3.1. Eigenvalue Analysis

Figure 3.3: Regions I, II and III for System II with $\theta = 0.95$ and $\epsilon = 0.0025$. The equilibrium is stable if the parameters are chosen inside Region I.

One pair of eigenvalues always remains in the open left half complex plane, i.e. Region III is never reached, because for this choice of parameters we have $d^2 > \epsilon$.

Figure 3.7 depicts again the evolution of the eigenvalues for System III under the influence of the parameter $q$. The parameter $d$ is again at $d = 0.1$, but $\epsilon$ is increased to $\epsilon = 0.25$ and the parameter $q$ is varied from $+\infty$ to $-\infty$. We expect that all four eigenvalue cross the imaginary axis at the same time, since for this case we have $d^2 < \epsilon$.

The arrows in the figure point in the direction of a decreasing value of $q$. For the limit $q \to +\infty$ (point 1), the eigenvalues are at $\{-\infty, 0, -\frac{d}{2} \pm i\sqrt{1 - \left(\frac{d}{2}\right)^2}\}$. The eigenvalues with vanishing imaginary part merge at $q = 2 - \frac{d}{d^2}(\text{point 2})$ and split again in a complex conjugated pair. At $q = d + 2\sqrt{\epsilon}$ (point 3), the complex conjugated pairs coincide. All eigenvalues cross the imaginary axis simultaneously at $q = -d$ (point 4), i.e. $\nu = 0$. Additional eigenvalue merging and splitting occur at $q = d - 2\sqrt{\epsilon}$ (point 5) and $q = -2 - \frac{d}{d^2}$ (point 6). For the $q$ tending to $-\infty$, three eigenvalues tend to the same location as for $q \to +\infty$ (point 1) and the fourth eigenvalue tends to $+\infty$. This figure qualitatively stays the same, if the values $d$ and $\epsilon$ are varied.
3.1. Eigenvalue Analysis

Figure 3.4: Regions I, II and III for System III with $\epsilon = 0.0025$. The equilibrium is stable if the parameters are chosen inside Region I.
3.1. Eigenvalue Analysis

Figure 3.5: Evolution of eigenvalues with positive imaginary part under the influence of $q$ for System II with $\theta = 0.95$, $d = 0.1$ and $\epsilon = 0.0025$. For $q = 0.03$ (point 1) and $q = 0$ (point 2), all eigenvalues are in the open left half complex plane. For $q = -0.03$ (point 4) one pair of eigenvalues has positive real part after crossing the imaginary axis at $\mu = 0$ (point 3).
3.1. Eigenvalue Analysis

Figure 3.6: Evolution of eigenvalues with positive imaginary part under the influence of $q$ for System III with $d = 0.1$ and $\epsilon = 0.0025$. For $q = 0.03$ (point 1) and $q = 0$ (point 2), all eigenvalues are in the open left half complex plane. For $q = -0.03$ (point 4) one pair of eigenvalues has positive real part after crossing the imaginary axis at $\mu = 0$ (point 3).
By increasing the value of $d$, the two circles on which the points 3,4 and 5 are located are shifted to the left. For a decreasing value of $\epsilon$, the two circles shrink until they vanish at $\epsilon = 0$. For $d$ large enough or $\epsilon$ small enough, i.e. for $d^2 > \epsilon$, the circles are completely inside the open left half complex plane and only one pair of eigenvalues cross the imaginary axis at $q = -\frac{\epsilon}{d}$, i.e. $\mu = 0$, as we have seen in Figure 3.6.

The analogous results for System II are shown in Figure 3.8 for $\theta = 0.95$. The parameters are again chosen again as $d = 0.1$ and $\epsilon = 0.25$. The eigenvalues for $q \to +\infty$ are at \{-\infty, 0, \ -\frac{d}{2} \pm i\theta - (\frac{d}{2})^2\} and they tend to \{\infty, 0, \ -\frac{d}{2} \pm i\theta - (\frac{d}{2})^2\} for $q \to -\infty$. The eigenvalue splitting as in Figure 3.7 at points 3 and 5 do not occur anymore and the eigenvalues cross the imaginary axis always sequentially. The first pair of eigenvalues becomes unstable at point 4a, whereas the second pair becomes unstable for a smaller value of $q$ at point 4b. If $d$ is chosen large enough or $\epsilon$ small enough, one pair of eigenvalues never crosses the imaginary axis and therefore Region III is never reached.

### 3.2 Nonlinear System

In order to determine the stability of the equilibrium it has been sufficient to investigate the Jacobian matrix. Yet, depending on the nonlinearities, the attractivity property is only valid in an (possibly arbitrarily small) neighborhood of the equilibrium. The region of attraction, or at least a lower bound, can be found using Lyapunov’s direct method.

**Theorem 3.2.1** (Lyapunov’s second (direct) method) Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$.

- If $\exists V(x)$ being LPDF and $\dot{V}(x) \leq 0 \ \forall x \in D$, then $x = 0$ is stable.
- If $\exists V(x)$ being LPDF and $-\dot{V}(x)$ is LPDF, then $x = 0$ is locally asymptotically stable.
- If $\exists V(x)$ being PDF and $-\dot{V}(x)$ is PDF, then $x = 0$ is globally asymptotically stable.

See [3] for the proof of Theorem 3.2.1.
3.2. Nonlinear System

Figure 3.7: Evolution of eigenvalues under the influence of $q$ for System III with $d = 0.1$ and $\epsilon = 0.25$. The value of $q$ is varied from $+\infty$ to $-\infty$. Eigenvalue merging and splitting occur at points 2, 3, 5 and 6. The eigenvalues cross the imaginary axis simultaneously at $\nu = 0$ (point 4).
3.2. Nonlinear System

Figure 3.8: Evolution of eigenvalues under the influence of $q$ for System II with $\theta = 0.95$, $d = 0.1$ and $\epsilon = 0.25$. The value of $q$ is varied from $+\infty$ to $-\infty$. Eigenvalue merging and splitting occur at points 2 and 6. The eigenvalues cross the imaginary axis sequentially first at $\mu = 0 \wedge \nu_1, \nu_2 > 0$ (point 4a) and then at $\mu = 0 \wedge \nu_1, \nu_2 < 0$ (point 4b).
3.2. Nonlinear System

Consider the following Lyapunov-function

\[ V_1(x) = \frac{1}{2}(\eta^2 + \dot{\eta}^2) + \frac{1}{2}\epsilon\theta(\theta\xi^2 + \dot{\xi}^2), \tag{3.11} \]

which is positive definite, since \( \epsilon > 0 \) and \( \theta > 0 \). The Lie derivative is defined as

\[ \dot{V}(x) = \nabla V(x)f(x) \quad \text{with} \quad \nabla V(x) := \left[ \frac{\partial V}{\partial x_1} \frac{\partial V}{\partial x_2} \frac{\partial V}{\partial x_3} \frac{\partial V}{\partial x_4} \right]. \tag{3.12} \]

Evaluating the Lie derivative of (3.11) for System II yields

\[ \dot{V}_1(x) = -(q + c_1\eta^2 + c_2\dot{\eta}^2)\dot{\eta}^2 - \epsilon\theta|d + \delta\xi|\dot{\xi}^2. \tag{3.13} \]

For \( q > 0 \) the derivative \( \dot{V}_1 \) is negative semi-definite. According to Theorem 3.2.1, the equilibrium is stable as we have expected from the eigenvalue analysis. Furthermore, we can determine the attractivity using LaSalle’s invariance principle.

**Theorem 3.2.1** (LaSalle’s invariance principle) Let \( x = 0 \) be an equilibrium point for \( \dot{x} = f(x) \). Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a positive definite function with \( \dot{V}(x) \leq 0 \ \forall \ x \in \mathbb{R}^n \). Let \( \Omega = \{ x \in \mathbb{R}^n \mid \dot{V}(x) = 0 \} \) and suppose that no solution can stay identically in \( \Omega \) except the trivial solution \( x(t) = 0 \). Then, \( x = 0 \) is globally asymptotically stable.

The proof of Theorem 3.2.1 is given in [3].

For (3.13) we have \( \Omega \subset \bar{\Omega} := \{ x \in \mathbb{R}^4 | \xi = 0 \} \). No solution can stay identically in \( \bar{\Omega} \) except for \( \eta = \xi = 0 \). According to the invariance principle 3.2.1, global attractivity of the equilibrium is guaranteed for \( q > 0 \).

The Lyapunov function \( V_1(x) \) proves global asymptotic stability of the equilibrium for System II and System III for \( q > 0 \), but it is inconclusive for \( q < 0 \). We first search for a Lyapunov function for System III.a, i.e. \( \delta = 0 \), with \( q < 0 \) and afterwards for System II.a with \( q < 0 \).

Consider the following positive definite Lyapunov candidate function

\[ V_2(x) = \frac{1}{2}((q\xi + \eta)^2 + (q\dot{\xi} + \dot{\eta})^2) + \frac{1}{2}\mu\left(\xi^2 + \dot{\xi}^2\right). \tag{3.14} \]

Evaluating the Lie derivative of \( V_2(x) \) for System III.a yields

\[ \dot{V}_2(x) = -(c_1\eta^2 + c_2\dot{\eta}^2)\left(\frac{q}{2}\dot{\xi} + \dot{\eta}\right)^2 - \left(\nu\mu - \left(\frac{q}{2}\right)^2(c_1\eta^2 + c_2\dot{\eta}^2)\right)\dot{\xi}^2, \tag{3.15} \]

which is only negative semidefinite inside the cylinder defined by \( \frac{\nu\mu}{q^2} = c_1\nu^2 + c_2\nu^2 \) and positive outside. Thus, only local asymptotic stability can be shown.
3.2. Nonlinear System

using LaSalle’s invariance principle. A global result can be found using the assumption $c = c_1 = c_2$. We add a nonlinear term to $V_2(x)$; that is

$$V_3(x) = V_2(x) + \frac{1}{4} \left( -\frac{q\epsilon}{\epsilon} \right) (\eta^2 + \dot{\eta}^2)^2,$$  \hspace{1cm} (3.16)

which is positive definite. Differentiating with respect to time yields

$$\dot{V}_3(x) = -\nu\mu\dot{\xi}^2 - \frac{c}{\epsilon} (\mu - q\nu) (\eta^2 + \dot{\eta}^2)\dot{\eta}^2 - \left( -\frac{qc^2}{\epsilon} \right) (\eta^2 + \dot{\eta}^2)^2\dot{\eta}^2, \hspace{1cm} (3.17)$$

which is globally negative semidefinite for $\mu \geq 0$ and $\nu > 0$. Using the invariance principle we can show global asymptotic stability for System III.a with $q < 0$ with the assumption $c = c_1 = c_2$.

For System II.a we consider the Lyapunov candidate function

$$V_4(x) = \frac{1}{2} \nu_1 \left( \eta + \theta q\xi + \frac{q(\theta - 1)}{\nu_1} \dot{\xi} \right)^2 + \frac{1}{2} \nu_1 \mu \dot{\xi}^2 + \frac{1}{2} \nu_2 \left( \dot{\eta} + \frac{q(\theta - 1)}{\nu_2} \dot{\xi} \right)^2 + \frac{1}{2} \nu_2 \mu \dot{\xi}^2 \hspace{1cm} (3.18)$$

with the corresponding Lie derivative

$$\dot{V}_4(x) = -\mu\dot{\xi}^2 - (c_1\eta^2 + c_2\dot{\eta}^2)(\nu_2\eta^2 - q(\theta - 1)\theta \dot{\eta} \xi + q\nu_2 \dot{\eta} \dot{\xi}). \hspace{1cm} (3.19)$$

With $V_4(x)$ we can show the asymptotic stability of the equilibrium of the linearization of System II, but it is inconclusive for the nonlinear case.
Chapter 4

Center Manifold Reduction

The equilibrium of System II is stable for $\mu > 0$, $\nu_1 > 0$ and $\nu_2 > 0$, as we have seen in Chapter 3. At $\mu = 0$, one eigenvalue pair cross the imaginary axis and the equilibrium becomes unstable. The equilibrium of System III is stable for $\mu > 0$ and $\nu > 0$. One pair of eigenvalues becomes unstable at $\mu = 0$ given that $d^2 > \epsilon$. If $d^2 < \epsilon$, all four eigenvalues cross the imaginary axis simultaneously at $\mu > 0$, since, for this case, the condition $\nu > 0$ is more restrictive. This special case occurs only if the eigenfrequency ration $\theta$ is exactly equal to unity. Thus, the case $\nu = 0$ is only of interest from a mathematical point of view and it is not considered any further.

Although the equilibrium becomes unstable for $\mu < 0$, the solutions still remain bounded due to the nonlinearities of the $\eta$-dynamics. This gives rise to a sub- or supercritical Hopf bifurcation under the influence of the bifurcation parameter $\mu$. One commonly used method to determine the normal form of the bifurcation, i.e. the simplest equation representing the type of bifurcation, is the center manifold reduction (CMR). In addition to the type of bifurcation, this method also yields an estimate of the periodic solution near the bifurcation. The CMR of the perfectly tuned case with a linear Helmholtz damper, i.e. System III.a, is presented in Section 4.1. The calculation is repeated for the more general System II.a, which is shown in Section 4.2.
4.1 Center Manifold Reduction of System III.a

The dynamics of System III.a, given by (2.77) and (2.79), can be written as

\[
\dot{x} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & -q & 0 & dq \\
0 & 0 & 0 & 1 \\
0 & 1 & -1 & -d
\end{pmatrix} A_0 x + \begin{pmatrix}
0 \\
-(c_1 x_1^2 + c_2 x_2^2)x_2 - \mu x_4 \\
0 \\
0
\end{pmatrix} g(x, \mu),
\] (4.1)

where \(\mu = \epsilon + dq\) is chosen as bifurcation parameter. The matrix \(A_0\) is the Jacobian of the System III.a for \(\mu = 0\) and the term \(g(x, \mu)\) contains the nonlinearities and the terms depending on \(\mu\).

The equilibrium of (4.1) is stable for \(\mu > 0\) and unstable for \(\mu < 0\), where we assume that \(\nu > 0\). The matrix \(A_0\) has one pair of eigenvalues \(\lambda_{1,2}\) with negative real part and one purely imaginary pair of eigenvalues \(\lambda_{3,4}\); that is

\[
\lambda_{1,2} = -\frac{\nu}{2} \pm i \sqrt{1 - \left(\frac{\nu}{2}\right)^2} \quad \text{and} \quad \lambda_{3,4} = \pm i.
\] (4.2)

The stable subspace, spanned by the eigenvectors corresponding to \(\lambda_{1,2}\), and the center subspace, spanned by the eigenvectors corresponding to \(\lambda_{3,4}\), are each 2-dimensional. We seek a linear transformation

\[
T^{-1} : x \rightarrow \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = T^{-1} x,
\] (4.3)

such that the dynamics in the new coordinates \(y\) and \(z\) can be written as

\[
\begin{pmatrix} y \\ z \end{pmatrix} = B \begin{pmatrix} y \\ z \end{pmatrix} + \tilde{g}(y, z, \mu),
\] (4.4)

where

\[
B = T^{-1} A_0 T = \begin{pmatrix} B_y & 0 \\ 0 & B_z \end{pmatrix} = \begin{pmatrix}
-\frac{\nu}{2} & \sqrt{1 - \left(\frac{\nu}{2}\right)^2} & 0 & 0 \\
\sqrt{1 - \left(\frac{\nu}{2}\right)^2} & -\frac{\nu}{2} & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\] (4.5)

\[
\tilde{g}(y, z, \mu) = T^{-1} g(Tx, \mu) = \begin{pmatrix}
\tilde{g}_y(y, z, \mu) \\
\tilde{g}_z(y, z, \mu)
\end{pmatrix}.
\] (4.6)
4.1. Center Manifold Reduction of System III.a

In order to find the matrix $T$, we define the matrices $T_A$ and $T_B$, whose column vectors consist of the eigenvectors of $A_0$ and $B$, respectively. Since the eigenvalues of $A_0$ and $B$ are identical, they have the same Jordan form $\Lambda$; thus we have

$$T_{A_0}^{-1}A_0T_{A_0} = \Lambda = T_B^{-1}BT_B$$

(4.7)

and the transformation $T$ becomes

$$T = T_{A_0}T_B^{-1} = \begin{pmatrix} q\sqrt{1-(\nu^2)^2} & \frac{1}{2}q\nu & -d & 0 \\ 0 & -q & 0 & d \\ -\sqrt{1-(\nu^2)^2} & -\frac{1}{2}\nu & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$  

(4.8)

Then, the term $\tilde{g}(y, z, \mu)$ in (4.4) is obtained as

$$\tilde{g}(y, z, \mu) = \left(c_1 \left(q\sqrt{1-(\nu^2)^2}y_1 + \frac{1}{2}q\nu y_2 - dz_1\right)^2(-qy_2 + dz_2) + c_2 (-qy_2 + dz_2)^3 + \mu(y_2 + z_2)\right) \begin{pmatrix} -\frac{1}{2}\sqrt{1-(\nu^2)^2} \\ \frac{1}{\nu} \\ 0 \\ -\frac{1}{\nu} \end{pmatrix}. \tag{4.9}$$

Since the new Jacobian matrix $B$ is chosen as a block-diagonal matrix, the dynamics of $y$ and $z$ are linearly decoupled and the coupling is only due to nonlinear terms and terms proportional to the bifurcation parameter $\mu$. No simplifications have been made up to this point and the dynamics in the new coordinates can be written as

$$\dot{y} = B_y y + \tilde{g}_y(y, z, \mu), \tag{4.10}$$

$$\dot{z} = B_z z + \tilde{g}_z(y, z, \mu), \tag{4.11}$$

$$\dot{\mu} = 0. \tag{4.12}$$

The bifurcation parameter $\mu$ is introduced as an additional state of the system. Since it is a constant parameter, its dynamics is given by $\dot{\mu} = 0$. This expansion is often referred to as ‘suspension trick’. The equation (4.10) describes the dynamics on the stable manifold and equation (4.11) describes the dynamics on the center manifold. The center manifold in an invariant manifold in the $(y, z, \mu)$-space with the center subspace as its tangent at the origin. Thus, the stability of the equilibrium at the bifurcation point is determined by the $z$-dynamics. The main idea of the center manifold reduction is to approximate the center manifold as $y = \pi(z, \mu)$ in the neighborhood of the equilibrium,
4.1. Center Manifold Reduction of System III.a

substitute this approximation into the system dynamics and obtain an approximation of the dynamics on the center manifold. In the transformed variables, the center subspace is described by \((y, \mu) = 0\). Since the center manifold is tangent to the center subspace, the Taylor series approximation of the CM has no constant terms or terms linear in \(z\) or \(\mu\). Therefore, we approximate the center manifold as

\[
y = \pi(z, \mu) = k_1 z_1^2 + k_2 z_2^2 + k_3 \mu^2 + k_4 z_1 z_2 + k_5 \mu z_1 + k_6 \mu z_2 + H.O.T. \tag{4.13}
\]

The total time derivative of (4.13) can be written as

\[
\dot{y} = \frac{\partial \pi}{\partial z} \dot{z} + \frac{\partial \pi}{\partial \mu} \dot{\mu} = \frac{\partial \pi}{\partial z} \dot{z}, \tag{4.14}
\]

where equation (4.12) was used. Substituting equations (4.10), (4.11) and (4.13) into (4.14) yields

\[
B_y \pi(z, \mu) + \tilde{g}_y(\pi(z, \mu), z, \mu) = \frac{\partial \pi}{\partial z} (B_z z + \tilde{g}_z(\pi(z, \mu), z, \mu)), \tag{4.15}
\]

where

\[
\frac{\partial \pi}{\partial z} = \begin{pmatrix} 2k_1 z_1 + k_4 z_2 + k_5 \mu \\ 2k_2 z_2 + k_4 z_1 + k_6 \mu \end{pmatrix}^T + H.O.T. \tag{4.16}
\]

The unknown coefficients \(\{k_1, \ldots, k_6\}\) of the approximation (4.13) can be found by evaluating (4.15) and equating coefficients of like terms; thus we have

\[
\begin{align*}
O(z_1^2) & : B_y k_1 & = k_4, \tag{4.17} \\
O(z_2^2) & : B_y k_2 & = -k_4, \tag{4.18} \\
O(\mu^2) & : B_y k_3 & = 0, \tag{4.19} \\
O(z_1 z_2) & : B_y k_4 & = -2k_1 + 2k_2, \tag{4.20} \\
O(\mu z_1) & : B_y k_5 & = k_6, \tag{4.21} \\
O(\mu z_2) & : B_y k_6 + \left(\frac{1}{2\sqrt{1-(\frac{1}{v})^2}} \frac{1}{v} \right)^T & = -k_5. \tag{4.22}
\end{align*}
\]
4.1. Center Manifold Reduction of System III.a

The solution to the linear system of equations (4.17)-(4.22) is

\[
\begin{align*}
    k_1 &= k_2 = k_3 = k_4 = 0, \\
    k_5 &= \left( \frac{1}{\nu^2 \sqrt{1 - (\frac{\nu}{2})^2}} \quad 0 \right)^T, \\
    k_6 &= \left( \frac{-1}{2 \nu^2 \sqrt{1 - (\frac{\nu}{2})^2}} \quad \frac{1}{\nu^2} \right)^T.
\end{align*}
\]

Therefore, the approximation of the center manifold is obtained as

\[
\begin{align*}
    \pi_1(z_1, z_2, \mu) &= \frac{1}{\nu^2 \sqrt{1 - (\frac{\nu}{2})^2}} \mu(z_1 - \frac{\nu}{2} z_2), \\
    \pi_2(z_1, z_2, \mu) &= \frac{1}{\nu^2} \mu z_2.
\end{align*}
\]

The dynamics on the center manifold is found by replacing \( y \) in (4.11) by the approximation of the center manifold (4.24)-(4.25) and neglecting terms of order \( \mathcal{O}(z^4) \) and higher with \( \mu = \mathcal{O}(\frac{1}{\nu}) \); that is

\[
\dot{z} = B z + \hat{g}_z(z, \mu),
\]

where

\[
\begin{align*}
    z &= (0 \quad 1 \quad 0)^T, \\
    \dot{z} &= \left( \frac{0}{\nu^2 \sqrt{1 - (\frac{\nu}{2})^2}} \mu(z_1 - \frac{\nu}{2} z_2) \right). \quad (4.26)
\end{align*}
\]

Writing (4.26) as one second order differential equation, we have

\[
\ddot{z}_1 + z_1 = -\frac{\mu}{\nu} \dot{z}_1 - \frac{d^3 c_1}{\nu} z_1^2 \dot{z}_1 - \frac{d^3 c_2}{\nu} z_3^3. \quad (4.27)
\]

As expected, the equilibrium of (4.27) is stable for \( \mu \geq 0 \) and unstable for \( \mu < 0 \). This can easily be seen using the Lyapunov function \( V = \frac{1}{2} z_1^2 + \frac{1}{2} z_3^2 \) whose Lie derivative given by \( \dot{V} = -\frac{\mu}{\nu} \dot{z}_1 - \frac{d^3 c_1}{\nu} z_1^2 \dot{z}_1 - \frac{d^3 c_3}{\nu} z_3^3 \). Since we consider only solutions near the equilibrium located at \( (z, \mu) = (0) \), we introduce a small parameter \( \kappa \), such that \( z_1 = \mathcal{O}(\kappa) \). The bifurcation parameter is chosen to be of order \( \mathcal{O}(\kappa^2) \) and we introduce a new variable \( u \) and define new parameters of order \( \mathcal{O}(1) \) as

\[
\begin{align*}
    u &= \frac{1}{\kappa} z_1, \\
    \hat{\mu} &= -\frac{1}{\kappa^2} \mu, \\
    \hat{c}_1 &= -\frac{d^3 c_1}{\nu}, \\
    \hat{c}_2 &= -\frac{d^3 c_2}{\nu}.
\end{align*}
\]

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4.1. Center Manifold Reduction of System III.a

Substituting (4.28)-(4.31) into (4.27) yields
\[ \ddot{u} + u = \kappa^2(\dot{\mu}\dot{u} + \hat{c}_1 u^2 \dot{u} + \hat{c}_2 \dot{u}^3). \] (4.32)

We introduce the transformation
\[ \zeta := \frac{1}{2}(u - i\dot{u}), \] (4.33)
\[ \bar{\zeta} := \frac{1}{2}(u + i\dot{u}), \] (4.34)

and rewrite the dynamics (4.32) in the new coordinate \( \zeta \) and \( \bar{\zeta} \) as
\[ \dot{\zeta} = i\zeta + \frac{1}{2}\kappa^2 \left( \hat{\mu}(\zeta - \bar{\zeta}) + \hat{c}_1(\zeta + \bar{\zeta})^2(\zeta - \bar{\zeta}) - \hat{c}_2(\zeta - \bar{\zeta})^3 \right). \] (4.35)

The unperturbed solution of (4.35) can be written as \( \zeta = re^{it} \). For a small perturbation, the solution is assumed to be of the same form, i.e. \( \zeta = re^{i\varphi} \), but with time-dependent parameters \( r \) and \( \varphi \). Substituting this Ansatz into (4.35) and keeping only the resonant terms\(^1\) yields
\[ \dot{r}e^{i\varphi} + i\dot{\varphi}re^{i\varphi} = ire^{i\varphi} + \frac{1}{2}\kappa^2 \left( \hat{\mu}re^{i\varphi} + \hat{c}_1r^3e^{i\varphi} + 3\hat{c}_2r^3e^{i\varphi} \right). \] (4.36)

A rigorous way of eliminating the non-resonant terms can be found in Appendix A. Separating real and imaginary parts of (4.36), we obtain
\[ \text{Re:} \quad \dot{r} = \frac{1}{2}\kappa^2\hat{\mu}r + \frac{1}{2}\kappa^2(\hat{c}_1 + 3\hat{c}_2)r^3, \] (4.37)
\[ \text{Im:} \quad \dot{\varphi} = 1. \] (4.38)

The equilibrium of (4.37) is stable for \( \hat{\mu} < 0 \) and unstable for \( \hat{\mu} > 0 \). At \( \hat{\mu} = 0 \) a pitchfork bifurcation occurs for the radius \( r \) and, thus, \( \zeta \) experiences a Hopf bifurcation. Since \( \hat{c}_1 + 3\hat{c}_2 < 0 \), the Hopf bifurcation is supercritical. The nontrivial equilibrium of (4.37) and (4.38) is
\[ r^* = \sqrt{-\frac{\hat{\mu}}{\hat{c}_1 + 3\hat{c}_2}}, \] (4.39)
\[ \varphi^* = t + \Delta\varphi. \] (4.40)

Applying the inverse of the transformation (4.33)-(4.34), the solution of \( u \) can

\(^1\)The resonant terms are the terms, which are proportional to the solution of the unperturbed solution, i.e. proportional to \( e^{it} \). The non-resonant terms can be eliminated using a near-identity transformation.
4.2 Center Manifold Reduction of System II.a

be written as

\[ u = \zeta + \bar{\zeta} = r^* (e^{i\varphi^*} + e^{-i\varphi^*}) = 2\sqrt{-\frac{\mu}{\hat{c}_1 + 3\hat{c}_2}} \cos (t + \Delta\varphi). \]  

(4.41)

It follows from (4.28)-(4.31) that

\[ z_1 = \kappa u = 2\sqrt{-\frac{\mu}{d^3(c_1 + 3c_2)}} \cos (t + \Delta\varphi). \]  

(4.42)

The solution of \( z_2 \) can be approximated by equation (4.11) as \( z_2 \approx -\dot{z}_1 \). The state \( y \) is assumed to be on the center manifold \( y = \pi(z, \mu) \). According to the approximation (4.24) and (4.25), we have

\[ y_1 = \frac{1}{\nu^2} \sqrt{1 - (\frac{\nu}{d})^2} \mu (z_1 + \frac{\nu}{2} \dot{z}_1), \]

(4.43)

\[ y_2 = -\frac{1}{\nu^2} \mu \dot{z}_1. \]

(4.44)

In order to find the dynamics in the original coordinates \( \eta \) and \( \xi \), we apply the transformation \( T \) defined by (4.3) and (4.8). Thus, the nontrivial solution in the original coordinates near the bifurcation is given by

\[ \eta = -(d - q \frac{\mu}{\nu^2})z_1 \approx -dz_1 = -2\sqrt{-\frac{\mu}{d (c_1 + 3c_2)}} \cos (t + \Delta\varphi), \]  

(4.45)

\[ \dot{\eta} = -(d - q \frac{\mu}{\nu^2}) \dot{z}_1 \approx -d \dot{z}_1 = 2\sqrt{-\frac{\mu}{d (c_1 + 3c_2)}} \sin (t + \Delta\varphi), \]  

(4.46)

\[ \xi = -(1 + \frac{\mu}{\nu^2})z_1 \approx -z_1 = -2\sqrt{-\frac{\mu}{d^3 (c_1 + 3c_2)}} \cos (t + \Delta\varphi), \]  

(4.47)

\[ \dot{\xi} = -(1 + \frac{\mu}{\nu^2}) \dot{z}_1 \approx -\dot{z}_1 = 2\sqrt{-\frac{\mu}{d^3 (c_1 + 3c_2)}} \sin (t + \Delta\varphi). \]  

(4.48)

The approximation of the periodic solution (4.45)-(4.48) is stable and it exists only for \( \mu < 0 \). We define \( r_\xi, r_\dot{\xi}, r_\eta \) and \( r_\dot{\eta} \) as the amplitude of oscillation of \( \xi, \dot{\xi}, \eta \) and \( \dot{\eta} \), respectively. The amplitudes are depicted in Figure 4.1 as a function of \( \mu \) scaled by the factor \( \frac{4}{d^3 (c_1 + 3c_2)} \). The solid line represents stable limit sets, whereas the dashed line represents unstable limit sets. The equilibrium of System III.a experiences a supercritical Hopf bifurcation at \( \mu = \epsilon + d\eta = 0 \).

4.2 Center Manifold Reduction of System II.a

The CMR of System II.a is conducted in the same way as for System III.a in Section 4.1. We rewrite the dynamics of System II.a, given by (2.73) and
4.2. Center Manifold Reduction of System II.a

Figure 4.1: The amplitude of oscillation of the states $\xi$, $\dot{\xi}$, $\eta$ and $\dot{\eta}$ near the Hopf bifurcation under the influence of the bifurcation parameter $\mu$. The solid (dashed) line represents stable (unstable) limit sets, respectively.
4.2. Center Manifold Reduction of System II.a

\[ (2.75), \text{as} \]

\[
\dot{x} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & -q & 0 & dq \\
0 & 0 & 0 & 1 \\
0 & 1 & -\theta & -d \\
\end{pmatrix} \begin{pmatrix}
0 \\
1 + (\theta - 1)^2 \\
0 \\
0 \\
\end{pmatrix} x + \begin{pmatrix}
0 \\
-(c_1 x_1^2 + c_2 x_2^2) x_2 - \mu x_4 \\
0 \\
0 \\
\end{pmatrix} g(x, \mu),
\]

(4.49)

where \( \mu = \epsilon \theta + dq \left( 1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2} \right) \) is chosen as bifurcation parameter. The matrix \( A_0 \) is the Jacobian of the System II.a for \( \mu = 0 \) and the term \( g(x, \mu) \) contains the nonlinearities and the terms depending on \( \mu \).

The equilibrium of (4.49) is stable for \( \mu > 0 \) and unstable for \( \mu < 0 \), where we assume that \( \nu_1 > 0 \) and \( \nu_2 > 0 \). The matrix \( A_0 \) has one pair of eigenvalues \( \lambda_{1,2} \) with negative real part and one purely imaginary pair of eigenvalues \( \lambda_{3,4} \); that is

\[
\lambda_{1,2} = -\frac{\nu_1}{2} \pm i \sqrt{\frac{\theta}{\omega^2} \left( \frac{\nu_1}{2} \right)^2} \quad \text{and} \quad \lambda_{3,4} = \pm i \omega,
\]

(4.50)

where

\[
\omega = \frac{\nu_2}{\nu_1}.
\]

(4.51)

We introduce new coordinates \( y = (y_1, y_2)^T \) and \( z = (z_1, z_2)^T \) with the corresponding linear transformation

\[
T : \begin{pmatrix} y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ \end{pmatrix} = T \begin{pmatrix} y \\ z \end{pmatrix},
\]

(4.52)

where

\[
T = \begin{pmatrix}
q \omega^2 \sqrt{\frac{\theta}{\omega^2} - \left( \frac{\nu_1}{2} \right)^2} & \frac{q}{\nu_1} (\frac{1}{2} \nu_1 \nu_2 - (\theta - 1)) & -d \omega & -\frac{d(\theta - 1)}{\nu_2} \\
-\frac{q}{\nu_1} \left( \theta - 1 \right) \sqrt{\frac{\theta}{\omega^2} - \left( \frac{\nu_1}{2} \right)^2} & -\frac{1}{2} q (1 + \theta) & -\frac{d(\theta - 1)}{\nu_2} d \omega & d \\
-\frac{q}{\nu_1} \omega^2 \sqrt{\frac{\theta}{\omega^2} - \left( \frac{\nu_1}{2} \right)^2} & -\frac{\nu_2}{\omega} & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

(4.53)
4.2. Center Manifold Reduction of System II.a

Rewriting the dynamics (4.49) in the new coordinates \( y \) and \( z \) yields

\[
\begin{pmatrix}
\dot{y} \\
\dot{z}
\end{pmatrix} = B \begin{pmatrix}
y \\
z
\end{pmatrix} + \tilde{g}(y, z, \mu),
\]

where

\[
B = T^{-1}A_0 T = \begin{pmatrix}
B_y & 0 \\
0 & B_z
\end{pmatrix} = \begin{pmatrix}
-\frac{\nu_1}{2} & -\sqrt{\frac{\theta}{\omega^2} - \left(\frac{\nu_1}{2}\right)^2} & 0 & 0 \\
\sqrt{\frac{\theta}{\omega^2} - \left(\frac{\nu_1}{2}\right)^2} & -\frac{\nu_1}{2} & 0 & 0 \\
0 & 0 & 0 & -\omega \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\tilde{g}(y, z, \mu) = T^{-1}g(Tx, \mu) = \begin{pmatrix}
\tilde{g}_y(y, z, \mu) \\
\tilde{g}_z(y, z, \mu)
\end{pmatrix}.
\]

Since the new Jacobian matrix \( B \) is a block-diagonal matrix, the dynamics of \( y \) and \( z \) are linearly decoupled and the coupling is only due to nonlinear terms and terms proportional to the bifurcation parameter \( \mu \). We introduce the bifurcation parameter \( \mu \) as an additional state of the system and write the dynamics as

\[
\begin{align*}
\dot{y} &= B_y y + \tilde{g}_y(y, z, \mu), \\
\dot{z} &= B_z z + \tilde{g}_z(y, z, \mu), \\
\dot{\mu} &= 0.
\end{align*}
\]

The equation (4.57) describes the dynamics on the stable manifold, whereas equation (4.58) describes the dynamics on the center manifold. Analogously to Section 4.1, we seek an approximation of the center manifold \( y = \pi(z, \mu) \) in the neighborhood of the equilibrium. Then, we substitute the approximation into (4.58) in order to find the dynamics on the center manifold. The stability of the \( z \)-dynamics will determine the stability of the equilibrium. The Ansatz for the center manifold is chosen as

\[
y = \pi(z, \mu) = k_1 z_1^2 + k_2 z_2^2 + k_3 \mu^2 + k_4 z_1 z_2 + k_5 \mu z_1 + k_6 \mu z_2 + \text{H.O.T.}
\]

Differentiating (4.60) w.r.t. time and substituting (4.57)-(4.59) yields

\[
B_y \pi(z, \mu) + \tilde{g}_y(\pi(z, \mu), z, \mu) = \frac{\partial \pi}{\partial z} (B_z z + \tilde{g}_z(\pi(z, \mu), z, \mu)),
\]

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4.2. Center Manifold Reduction of System II.a

where

\[ \frac{\partial \pi}{\partial z} = \left(2k_1z_1 + k_4z_2 + k_5\mu \right)^T + \text{H.O.T.} \tag{4.62} \]

Equating coefficients of like terms in (4.61) we obtain

\[ O(z_1^2) : \quad B_yk_1 = \omega k_4, \]
\[ O(z_2^2) : \quad B_yk_2 = -\omega k_4, \]
\[ O(\mu^2) : \quad B_yk_3 = 0, \]
\[ O(z_1z_2) : \quad B_yk_4 = -2\omega k_1 + 2\omega k_2, \]
\[ O(\mu z_1) : \quad B_yk_5 = \omega k_6, \]
\[ O(\mu z_2) : \quad B_yk_6 + \hat{h} = -\omega k_5, \]

where

\[ \hat{h} = \left( -\frac{1}{\sqrt{\frac{\nu_1\nu_2}{\nu_1^2\nu_2^2 - (d^2 - q^2\theta)^2}}} \left(h_1 \frac{\nu_1\nu_2}{\nu_1^2\nu_2^2 - (d^2 - q^2\theta)^2} \right) \right), \tag{4.69} \]
\[ h_1 = \frac{(\theta - 1)\nu_1\nu_2\omega(d^2 - q^2\theta)}{\nu_1^2\nu_2^2 + (\theta - 1)^2(d^2 - q^2\theta)^2}, \tag{4.70} \]
\[ h_2 = -\frac{\nu_1^2\nu_2^2}{\nu_1^2\nu_2^2 + (\theta - 1)^2(d^2 - q^2\theta)^2}. \tag{4.71} \]

The unknown coefficients \( \{k_1, \ldots, k_6\} \) of the approximation (4.60) are found by solving the linear system of equations (4.63)-(4.68); that is

\[ k_1 = k_2 = k_3 = k_4 = 0, \]
\[ k_5 = -\omega [B_y^2 + \omega^2 I]^{-1}\hat{h}, \]
\[ k_6 = -B_y[B_y^2 + \omega^2 I]^{-1}\hat{h}. \tag{4.72} \]

Thus, the approximation of the center manifold is

\[ \pi(z_1, z_2, \mu) = k_5\mu z_1 + k_6\mu z_2. \tag{4.73} \]

The dynamics on the center manifold is found by replacing \( y \) in (4.58) by the approximation of the center manifold (4.73). After neglecting higher order terms we obtain

\[ \dot{z} = B_z z + \dot{g}_z(\pi(z, \mu), z, \mu) \approx \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} z + \begin{pmatrix} \mu z_2 + b_1z_1^3 + b_2z_1^2z_2 + b_3z_1z_2^2 + b_4z_2^3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{4.74} \]
4.2. Center Manifold Reduction of System II.a

where

\[ b_1 = -\frac{d^3(\theta - 1)\omega}{\nu_1\nu_2^2} (c_1\nu_1^2 + c_2(\theta - 1)) , \]  
(4.75)

\[ b_2 = \frac{d^3}{\nu_1\nu_2^2} (c_1\nu_1(\nu_1\nu_2 - 2(\theta - 1)^2) + 3c_2\nu_2(\theta - 1)^2) , \]  
(4.76)

\[ b_3 = -\frac{d^3}{\nu_2^2}(\theta - 1)\omega (c_1 (-2\nu_1\nu_2 + (\theta - 1)^2) + 3c_2\nu_2^2) , \]  
(4.77)

\[ b_4 = \frac{d^3}{\nu_2^2} (c_1(\theta - 1)^2 + c_2\nu_2^2) \]  
(4.78)

and \( h_1 \) and \( h_2 \) defined by (4.70) and (4.71). Differentiating the first equation of (4.74) w.r.t. time yields

\[ \ddot{z}_1 = -\omega \dot{z}_2 + h_1 \left( \mu \dot{z}_2 + 3b_1 z_1^2 \dot{z}_1 + b_2 (2z_1 z_2 \dot{z}_1 + z_1^2 \dot{z}_2) \right. \]
\[ \left. + b_3(z_2^2 \dot{z}_1 + 2z_1 z_2 \dot{z}_2) + 3b_4 z_2^2 \dot{z}_2 \right) . \]  
(4.79)

As first order approximation, we can write \( z_2 \) and its derivative as \( z_2 \approx -\frac{1}{\omega} \dot{z}_1 \) and \( \dot{z}_2 \approx \omega z_1 \). Substituting the second equation of (4.74) into (4.79) and omitting higher order terms, we obtain a second order differential equation in \( z_1 \); that is

\[ \ddot{z}_1 + \omega^2 z_1 = \omega h_1 \mu z_1 + h_2 \mu \dot{z}_1 + \omega (b_2 h_1 - b_1 h_2) z_1^2 + ((3b_1 - 2b_2)h_1 + b_2 h_2) z_1^2 \dot{z}_1 \]
\[ + \frac{1}{\omega} ((-2b_2 + 3b_4)h_1 - b_3 h_2) z_1 \dot{z}_1^2 + \frac{1}{\omega^2} (b_3 h_1 + b_4 h_2) \dot{z}_1^3 . \]  
(4.80)

The stability of the equilibrium of (4.80) for \( \mu = 0 \) can not be determined by the linear terms. Thus, we seek the normal form describing the dynamics near the equilibrium located at \( (z, \mu) = 0 \). We introduce a small parameter \( \kappa \), such that \( z_1 = O(\kappa) \). The bifurcation parameter is chosen to be of order \( O(\kappa^2) \) and
4.2. Center Manifold Reduction of System II.a

we define a new variable \( u \) and new parameters of order \( O(1) \) as

\[
\begin{align*}
    u &= \frac{1}{\kappa}z_1, \\
    \mu_1 &= \frac{1}{\kappa^2}\omega h_1 \mu, \\
    \mu_2 &= \frac{1}{\kappa^2}h_2 \mu, \\
    a_1 &= \omega (b_2 h_1 - b_1 h_2), \\
    a_2 &= ((3b_1 - 2b_3)h_1 + b_2 h_2), \\
    a_3 &= \frac{1}{\omega} ((-2b_2 + 3b_4)h_1 - b_3 h_2), \\
    a_4 &= \frac{1}{\omega^2} (b_3 h_1 + b_4 h_2).
\end{align*}
\]

Using these definitions, (4.80) becomes

\[
\ddot{u} + \omega^2 u = \kappa^2 (\mu_1 u + \mu_2 \dot{u} + a_1 u^3 + a_2 u^2 \dot{u} + a_3 u a^2 + a_4 \dot{u}^3).
\]

Applying the method of normal forms (see Appendix A), we find that the dynamics of \( u \) experiences a Hopf bifurcation under the influence of the bifurcation parameter \( \mu \). The periodic solution near the bifurcation can be written as

\[
u = 2r_u^* \cos \varphi_u^*.
\]

The radius \( r_u^* \) and the angle \( \varphi_u^* \) are given by

\[
\begin{align*}
    r_u^* &= \sqrt{-\frac{\mu_2}{a_2 + 3\omega^2 a_4}}, \\
    \varphi_u^* &= \omega \left( 1 - \frac{\kappa^2}{2\omega^2} \left( \frac{\mu_1 - 3a_1 + \omega^2 a_3}{a_2 + 3\omega^2 a_4} \mu_2 \right) \right) t + \Delta \varphi,
\end{align*}
\]

where the phase \( \Delta \varphi \) is arbitrary. This ‘freedom of phase’ occurs, because we are dealing with an autonomous system. It follows from (4.81) that the periodic solution of \( z_1 \) is

\[
z_1 = r^* \cos \varphi^*,
\]
4.2. Center Manifold Reduction of System II.a

with

\[ r^* = 2Kr^*_{u} = \sqrt{-\frac{4\mu h_2}{(3b_1 + b_3)h_1 + (b_2 + 3b_4)h_2}}, \]  

(4.93)  

\[ \varphi^* = \varphi^*_{u} = \omega \left( 1 - \frac{1}{2\omega^\mu} \left( \frac{(3b_1 + b_3)(h_1^2 + h_2^2)}{(3b_1 + b_3)h_1 + (b_2 + 3b_4)h_2} \right) \right) t + \Delta \varphi. \]  

(4.94)

Substituting the definitions (4.70), (4.71) and (4.75)-(4.78) yields

\[ r^* = \sqrt{-\frac{4\mu}{d^3 \left( \nu_1 c_1 + 3\nu_2 \right) \left( 1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2} \right) \left( 1 + \frac{(\theta - 1)^2(d^2 - q^2\theta)}{\nu_1^2 \nu_2} \right) + \nu_1 \nu_2}}, \]  

(4.95)  

\[ \varphi^* = \omega \left( 1 - \frac{1}{2\omega^\mu} \left( \frac{(\theta - 1)\nu_1 \nu_2}{\nu_1^2 \nu_2 + (\theta - 1)^2(d^2 - q^2\theta)} \right) \right) t + \Delta \varphi. \]  

(4.96)

Note that the denominator in (4.95) is always positive, since \( \nu_1 > 0, \nu_2 > 0 \) and \( d^2 - q^2\theta = \nu_1 \nu_2 - q(\nu_1 \theta + \nu_2) > 0 \). Because, additionally, the equilibrium is unstable for \( \mu < 0 \), the Hopf bifurcation is supercritical. According to (4.96), the eigenfrequency of oscillation at \( \mu = 0 \) is equal to the imaginary part of the purely imaginary eigenvalues \( \lambda_{3,4} \). Yet, unlike System III.a, the frequency of oscillation near the bifurcation depends on the bifurcation parameter \( \mu \).

The solution of \( z_2 \) can be approximated with equation (4.74) as \( z_2 \approx -\frac{1}{\omega} \dot{z}_1 \), thus we have

\[ z_2 = r^* \sin \varphi^*. \]  

(4.97)

The state \( y \) is assumed to be on the center manifold \( y = \pi(z, \mu) \). According to the approximation (4.73) we have

\[ y = k_5 \mu z_1 + k_6 \mu z_2. \]  

(4.98)

In order to find the dynamics in the original coordinates \( \eta \) and \( \xi \), we apply the transformation \( T \) defined by (4.52) and (4.53). By omitting higher order terms\(^2\), the nontrivial solution in the original coordinates near the bifurcation

\(^2\)The state \( y \) can be omitted, since it is of higher order than the state \( z \).
4.2. Center Manifold Reduction of System II.a

is given by

\[ \eta \approx -\frac{d}{\omega} z_1 - \frac{d(\theta - 1)}{\nu_2} z_2 = -\frac{d}{\omega} \sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}} r^* \cos (\varphi^* + \Delta \phi), \quad (4.99) \]
\[ \dot{\eta} \approx -\frac{d(\theta - 1)}{\nu_1 \omega} z_1 + d z_2 = d \sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}} r^* \sin (\varphi^* + \Delta \phi), \quad (4.100) \]
\[ \xi \approx -\frac{1}{\omega} z_1 = -\frac{1}{\omega} r^* \cos \varphi^*, \quad (4.101) \]
\[ \dot{\xi} \approx z_2 = r^* \sin \varphi^*, \quad (4.102) \]

where

\[ \Delta \phi = \arccos \left( \frac{1}{\sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}}} \right). \quad (4.103) \]

The approximation of the periodic solution (4.99)-(4.102) is stable and it exists only for \( \mu < 0 \). For System III.a, the oscillation of \( \eta \) and \( \xi \) are in phase, whereas we have a phase shift of \( \Delta \phi \) between \( \eta \) and \( \xi \) for System II.a.
Chapter 5

Method of Multiple Scales

The method of multiple scales is a technique for approximating the solution of problems with weak nonlinearities. The main idea is to introduce fast and slow timescales and split the solution into fast-scale and slow-scale components. It is assumed that the dynamics on these different timescales are independent and that the solution does not contain secular terms, i.e. terms growing unbounded with time.

In Section 5.1 the method of multiple scales is shown theoretically for a system experiencing a Hopf bifurcation. The method is applied to System II.a in Section 5.2 and the results for System III.a can easily be obtained because it is a special case of System II.a.

5.1 Theory

In this section it is shown how to compute the normal form of a Hopf bifurcation using the method of multiple scales. The derivation can be found in [6].

We consider a nonlinear autonomous system of the form

\[ \dot{x} = f(x, \mu), \quad (5.1) \]

which experiences a Hopf bifurcation under the influence of a bifurcation parameter \( \mu \). By expanding (5.1) for small \( x \) and \( \mu \), we obtain

\[ \dot{x} = A_0 x + b \mu + \mu B x + Q(x, x) + C(x, x, x) + \text{H.O.T.}, \quad (5.2) \]

where \( A_0 = \frac{\partial f}{\partial x} \), \( b = \frac{\partial f}{\partial \mu} \) and \( B = \frac{\partial^2 f}{\partial x \partial \mu} \). The summands \( Q(x, x) \) and \( C(x, x, x) \) contain the quadratic and cubic terms, respectively. They are homogeneous
5.1. Theory

in their arguments, i.e. they can be written as

\[ Q(x, x) = \sum_{i,j} \alpha_{ij} x_i x_j, \quad (5.3) \]

\[ C(x, x, x) = \sum_{k,l,m} \beta_{klm} x_k x_l x_m. \quad (5.4) \]

The matrix \( \alpha \) and the tensor \( \beta \) can be chosen symmetric, i.e. \( \alpha_{ij} = \alpha_{ji} \) and \( \beta_{klm} = \beta_{kml} = \beta_{lk} = \beta_{lmk} = \beta_{mlk} \). The matrix \( A_0 \) has one purely imaginary pair of eigenvalues and all other eigenvalues are in the open left half complex plane. Without loss of generality\(^1\), we assume that the vector \( b_\mu \) is not present in (5.2) and therefore we have

\[ \dot{x} = A_0 x + \mu B x + Q(x, x) + C(x, x, x) + H.O.T. \quad (5.5) \]

Near the Hopf bifurcation, i.e. for small \( \mu \), the solution \( x \) is small. The smallness of \( x \) is captured by a small parameter \( \kappa \) and we introduce a new state \( y \) as

\[ x = \kappa y. \quad (5.6) \]

Substituting (5.6) into (5.5) yields

\[ \dot{y} = A_0 y + \mu B y + \kappa Q(y, y) + \kappa^2 C(y, y, y) + O(\kappa^3). \quad (5.7) \]

We note that the nonlinearities in (5.7) are small because we consider solutions near the Hopf bifurcation.

The experience gained by using other perturbation techniques has shown that the functional dependence of the state \( y \) on \( t \) and \( \kappa \) is not disjoint. Besides the individual \( t \) and \( \kappa \), it also depends on the combinations \( \{\kappa t, \kappa^2 t, \kappa^3 t, \ldots \} \).

This motivates us to introduce new timescales as

\[ T_i = \kappa^i t, \quad \text{for } i = \{0, 1, 2, \ldots \}. \quad (5.8) \]

We note that \( T_0 \) represents a fast timescale, \( T_1 \) represents a slow timescale, \( T_2 \) represents an even slower timescale, and so on. The total time derivative becomes

\[ \frac{d}{dt} = \sum_{i=0}^{\infty} \kappa^i D_i, \quad \text{where} \quad D_i = \frac{d}{dT_i}. \quad (5.9) \]

\(^1\)This can always be achieved by shifting the fixed point by \( -A_0^{-1} b_\mu \), since \( A_0 \) is non-singular.
5.1. Theory

We seek a uniform expansion of the solution of the form

\[ y(T_0, T_1, T_2, \ldots) = \sum_{i=0}^{\infty} \kappa^i y_i(T_0, T_1, T_2, \ldots). \]  

(5.10)

The bifurcation parameter \( \mu \) is also expanded in \( \kappa \); that is

\[ \mu = \sum_{i=0}^{\infty} \kappa^i \mu_i. \]  

(5.11)

Substituting (5.8)-(5.11) into (5.7) and equating terms of the same order of \( \kappa \) yields

\[ \mathcal{O}(1) : (D_0 - (A_0 + \mu_0 B)) y_0 = 0, \]  

(5.12)

\[ \mathcal{O}(\kappa) : (D_0 - (A_0 + \mu_0 B)) y_1 = (\mu_1 B - D_1) y_0 + \mathcal{Q}(y_0, y_0), \]  

(5.13)

\[ \mathcal{O}(\kappa^2) : (D_0 - (A_0 + \mu_0 B)) y_2 = (\mu_2 B - D_2) y_0 + \mu_1 B y_1 - D_1 y_1 + 2\mathcal{Q}(y_0, y_1) + \mathcal{C}(y_0, y_0, y_0). \]  

(5.14)

To keep the expansion (5.10) uniform, the secular terms in \( y_0, y_1 \) and \( y_2 \) must be eliminated. According to (5.12), we choose \( \mu_0 = 0 \) in order to avoid secular terms in \( y_0 \). Therefore, the longterm behaviour of the solution to (5.12) is governed by the purely imaginary eigenvalues and the terms corresponding to the eigenvalues with negative real part decay with time, which yields

\[ y_0 = z(T_1, T_2, \ldots) \mathbf{p} e^{i\omega T_0} + \bar{z}(T_1, T_2, \ldots) \bar{\mathbf{p}} e^{-i\omega T_0}. \]  

(5.15)

The vector \( \mathbf{p} \) in (5.15) is the eigenvector of \( A_0 \) corresponding to the eigenvalue \( i\omega \); that is

\[ A_0 \mathbf{p} = i\omega \mathbf{p}. \]  

(5.16)

Substituting (5.15) into (5.13) yields

\[ (D_0 - A_0) y_1 = (\mu_1 B - D_1) z \mathbf{p} e^{i\omega T_0} + \mathcal{Q}(\mathbf{p}, \mathbf{p}) z^2 e^{2i\omega T_0} + \mathcal{Q}(\mathbf{p}, \bar{\mathbf{p}}) \bar{z} + \mathcal{C}, \]  

(5.17)

where the acronym \( \mathcal{C} \) denotes ‘complex conjugate’. Define \( \mathbf{q} \) as the left eigenvector of \( A_0 \) corresponding to the eigenvalue \( i\omega \), i.e. \( A_0^T \mathbf{q} = i\omega \mathbf{q} \), and scale it such that \( \mathbf{q}^T \mathbf{p} = 1 \). In order that \( y_1 \) does not contain secular terms, the term in (5.17) proportional to \( e^{i\omega T_0} \) has to be perpendicular to \( \mathbf{q} \) (see Appendix B), which gives us the following condition on \( z \):

\[ (D_1 - \mu_1 \mathbf{q}^T \mathbf{B} \mathbf{p}) z = 0. \]  

(5.18)
5.1. Theory

Additionally, the term of (5.17) proportional to $e^{-i\omega T_0}$ has to be perpendicular to $\bar{q}$, yet this yields the same result as (5.18). For $\mu_1 q^T B p \neq 0$, the coefficient $z$ grows unbounded either for $\mu_1 < 0$ or for $\mu_1 > 0$ and the expansion (5.10) is not uniform. Therefore, we choose $\mu_1 = 0$ and equation (5.18) becomes $D_1 z = 0$, i.e. $z$ is not a function of $T_1$. Then, (5.17) reduces to

$$\left(D_0 - A_0\right) y_1 = Q(p, p) z^2 e^{2i\omega T_0} + Q(p, \bar{p}) \bar{z}^2 + cc. \quad (5.19)$$

The homogeneous solution of (5.19) is omitted, since it is already covered by $y_0$. The particular solution can be written as

$$y_1 = \zeta_2 z^2 e^{2i\omega T_0} + \zeta_0 \bar{z}^2 + cc, \quad (5.20)$$

where

$$\zeta_2 = \left[2i\omega I - A_0\right]^{-1} Q(p, p), \quad \zeta_0 = -A_0^{-1} Q(p, \bar{p}). \quad (5.21)$$

Since the only eigenvalues of $A_0$ with vanishing real part are at $\pm i\omega$, the matrices in (5.21) are regular and $\zeta_2$ and $\zeta_0$ are uniquely defined. Substituting (5.15), (5.20) and (5.21) into (5.14) and using $\mu_0 = \mu_1 = 0$ and $D_1 y_1 = 0$, we obtain

$$\left(D_0 - A_0\right) y_2 = (-D_2 p z + \mu_2 B p z + 4Q(p, \zeta_0) z^2 \bar{z} + 2Q(\bar{p}, \zeta_2) \bar{z}^2 \bar{z} + 3C(p, p, \bar{p}) \bar{z}^2 \bar{z}) e^{i\omega T_0} + cc + nst. \quad (5.22)$$

The acronym $nst$ denotes ‘non-secular terms’ and captures all terms, which do not generate secular terms. In (5.22) the non-secular terms are terms proportional to $e^{3i\omega T_0}$ and $e^{-3i\omega T_0}$. Analogously to $y_1$, the solution $y_2$ must not contain secular terms, which is achieved by choosing the term in (5.22) proportional to $e^{i\omega T_0}$ perpendicular to $q$. Imposing this condition, we obtain the following condition for $z(T_2)$:

$$D_2 z = \alpha \mu_2 z + \beta z^2 \bar{z}, \quad (5.23)$$

where

$$\alpha = q^T B p, \quad (5.24)$$

$$\beta = 4q^T Q(p, \zeta_0) + 2q^T Q(\bar{p}, \zeta_2) + 3q^T C(p, p, \bar{p}). \quad (5.25)$$

Using $\mu \approx \kappa^2 \mu_2$, the total time derivative of $z(T_2, \ldots)$ becomes

$$\frac{d}{dt} z = D_0 z + \kappa D_1 \bar{z}^0 + \kappa^2 D_2 z + \cdots \approx \alpha \mu z + \kappa^2 \beta z^2 \bar{z}. \quad (5.26)$$

The solution $x$ of the original system (5.1) can be approximated using the first
5.1. Theory

term of the uniform expansion (5.10) as
\[ x = \kappa y \approx \kappa y_0 = z_x p e^{i\omega t} + cc, \]  
(5.27)

where
\[ z_x = \kappa z. \]  
(5.28)

It follows from (5.26) that the dynamics of \( z_x \) is given by
\[ \frac{d}{dt} z_x = \alpha \mu z_x + \beta z_x^2 \bar{z}_x. \]  
(5.29)

Writing \( z_x \) in polar coordinates, i.e. \( z_x(t) = r(t)e^{i\varphi(t)} \), substituting into (5.29) and separating real and imaginary parts, we obtain
\[ \text{Re: } \dot{r} = \mu \alpha_R r + \beta_R r^3, \]  
(5.30)
\[ \text{Im: } \dot{\varphi} = \mu \alpha_I + \beta_I r^2, \]  
(5.31)

where \( \alpha_R \) and \( \alpha_I \) are the real and imaginary parts of \( \alpha \), respectively and \( \beta_R \) and \( \beta_I \) are the real and imaginary parts of \( \beta \), respectively. The non-trivial equilibrium of (5.30) is
\[ r^* = \sqrt{-\frac{\alpha_R}{\beta_R}}, \]  
(5.32)

and it exists only for \( \mu \frac{\alpha_R}{\beta_R} < 0 \). The corresponding angle \( \varphi^* \) is found as
\[ \varphi^*(t) = \mu \frac{\alpha_I \beta_R - \alpha_R \beta_I}{\beta_R} t + \Delta \varphi, \]  
(5.33)

where the phase \( \Delta \varphi \) is arbitrary. Therefore, the coefficient \( z_x \) becomes
\[ z_x = r^* e^{i\varphi^*}. \]  
(5.34)

Substituting (5.34) into (5.27), the approximation of the periodic solution of (5.1) near the Hopf bifurcation is found as
\[ x(t) \approx r^* \left( p e^{i(\omega t + \varphi^*)} + \bar{p} e^{-i(\omega t + \varphi^*)} \right). \]  
(5.35)

Using Euler’s formula, (5.35) can be written as
\[ x(t) \approx 2r^* \left( p_R \cos (\omega t + \varphi^*) - p_I \sin (\omega t + \varphi^*) \right), \]  
(5.36)

where \( p_R \) and \( p_I \) are the real and imaginary parts of the eigenvector \( p \), respectively.
5.2 Method of Multiple Scales applied to System II.a and System III.a

In this Section we investigate System II.a and System III.a using the method of multiple scales as it is described in Section 5.1. We apply the method only to the more general System II.a, since System III.a is a special case of System II.a.

We rewrite the dynamics of System II.a, given by (2.73) and (2.75), such that it is of the form (5.5); that is

\[
\dot{x} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & -q & 0 & dq \left(1 + \frac{(\theta-1)^2}{\nu_1 \nu_2}\right) \\
0 & 0 & 0 & 1 \\
0 & 1 & -\theta & -d
\end{pmatrix} \begin{pmatrix} x \\ A_0 \end{pmatrix} + \mu \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix} x \\ B \end{pmatrix} + \begin{pmatrix}
0 \\
-(c_1 x_1^2 + c_2 x_2^2) x_2 \\
0 \\
0
\end{pmatrix},
\]

(5.37)

where \( \mu = \epsilon \theta + dq \left(1 + \frac{(\theta-1)^2}{\nu_1 \nu_2}\right) \) is chosen as bifurcation parameter. As we have have seen in Section 4.2, the Jacobian matrix \( A_0 \) has one pair of eigenvalues with negative real part, i.e. \( \lambda_{1,2} = -\frac{\nu_2}{2} \pm i \sqrt{\frac{\theta}{2} - \left(\frac{\nu_2}{2}\right)^2} \), and one purely imaginary pair of eigenvalues \( \lambda_{3,4} = \pm i \omega \), where \( \omega = \frac{\nu_2}{\nu_1} \). The vector \( Q(x,x,x) \) does not appear in (5.37), since the dynamics does not contain any quadratic terms. In Section 5.1, the vector \( C(x,x,x) \) is assumed to be homogeneous in its arguments. Therefore, it is written as

\[
C(x,x,x) = \sum_{k,l,m} b_{klm} x_k x_l x_m,
\]

(5.38)

where

\[
b_{112} = b_{121} = b_{211} = \begin{pmatrix}
0 \\
-\frac{1}{3} c_1 \\
0
\end{pmatrix}, \quad b_{222} = \begin{pmatrix}
0 \\
-c_2 \\
0
\end{pmatrix}.
\]

(5.39)
5.2. Method of Multiple Scales applied to System II.a and System III.a

The right eigenvector \( \mathbf{p} \) of \( \mathbf{A}_0 \) corresponding to the eigenvalue \( i \omega \) is calculated as

\[
\mathbf{p} = \begin{pmatrix}
-\frac{d}{\omega} \frac{\theta - 1}{\sqrt{\nu_1 \nu_2}} \\
d \\
0 \\
1
\end{pmatrix} - i \begin{pmatrix}
\frac{d}{\frac{\theta - 1}{\sqrt{\nu_1 \nu_2}}} \\
\frac{1}{\omega} \\
0
\end{pmatrix}.
\]

(5.40)

The left eigenvector \( \mathbf{q} \) of \( \mathbf{A}_0 \) corresponding to the eigenvalue \( i \omega \), scaled such that \( \mathbf{q}^T \mathbf{p} = 1 \), is found to be

\[
\mathbf{q} = \frac{1}{2} \frac{1}{\nu_1^3 \nu_2^3 + (\theta - 1)^2(d^2 - q^2 \theta)^2} \begin{pmatrix}
-\nu_1 \nu_2 (\theta - 1)(d^2 - q^2 \theta) \\
\nu_1^2 \nu_2^3 \\
-\nu_1 \nu_2 (\theta - 1)(d^2 - q^2 \theta) \\
\nu_1 \nu_2 (\theta - 1)(d^2 - q^2 \theta)
\end{pmatrix}
\]

\[
+ i \frac{1}{2} \frac{1}{\nu_1^3 \nu_2^3 + (\theta - 1)^2(d^2 - q^2 \theta)^2} \begin{pmatrix}
\omega \nu_1 \nu_2 (\theta - 1)(d^2 - q^2 \theta) \\
\nu_1 \nu_2 (\theta - 1)(d^2 - q^2 \theta)
\end{pmatrix}.
\]

(5.41)

Using the calculated left eigenvector \( \mathbf{p} \), the vector \( \mathbf{C}(\mathbf{p}, \mathbf{p}, \bar{\mathbf{p}}) \) becomes

\[
\mathbf{C}(\mathbf{p}, \mathbf{p}, \bar{\mathbf{p}}) = \begin{pmatrix}
0 \\
-\frac{1}{3} c_1 (p_1^2 \bar{p}_2 + 2 p_1 \bar{p}_1 p_2) - c_2 p_2^2 \bar{p}_2 \\
0 \\
0
\end{pmatrix}
\]

\[
= -\frac{1}{3} d^3 \left( \frac{1}{\nu_1 \nu_2} \right) \left( \frac{\nu_1}{\nu_2} c_1 + 3 c_2 \right) \left( 1 - i \frac{\theta - 1}{\sqrt{\nu_1 \nu_2}} \right) \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}.
\]

(5.42)

According to (5.36), the periodic solution of \( \mathbf{x} \) near the Hopf bifurcation is given by

\[
\mathbf{x}(t) \approx 2r^* \left( \mathbf{p}_R \cos(\omega t + \varphi^*) - \mathbf{p}_I \sin(\omega t + \varphi^*) \right).
\]

(5.43)

The radius \( r^* \) and the angle \( \varphi^* \) are given by (5.32) and (5.33). Additionally,
5.2. Method of Multiple Scales applied to System II.a and System III.a

using (5.24) and (5.25) yields

\[ r^* = \sqrt{-\mu \frac{\alpha_R}{\beta_R}}, \] (5.44)

\[ \varphi^* = \mu \frac{\alpha_I \beta_R - \alpha_R \beta_I}{\beta_R} t + \Delta \varphi, \] (5.45)

where

\[ \alpha_R = \Re(q^T B p), \quad \beta_R = \Re(3q^T C (p, p, \bar{p})), \]
\[ \alpha_I = \Im(q^T B p), \quad \beta_I = \Im(3q^T C (p, p, \bar{p})). \] (5.46)

Substituting (5.40)-(5.42) into (5.44)-(5.46) yields

\[ r^* = \sqrt{-\mu \frac{c_1 + 3c_2}{\nu_1 \nu_2} \left( 1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2} \right) \left( 1 + \frac{(\theta - 1)^2 (d^2 - q^2 \theta)}{\nu_1 \nu_2} \right)}, \] (5.47)

\[ \varphi^* = -\frac{1}{2} \omega \mu \frac{(\theta - 1) \nu_1 \nu_2}{\nu_1 \nu_2 + (\theta - 1)^2 (d^2 - q^2 \theta)} t + \Delta \varphi. \] (5.48)

The periodic solution in the original coordinates \( \eta \) and \( \xi \) are obtained by substituting (5.40) into (5.43); that is

\[ \eta = \frac{d}{\omega} \sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}} 2r^* \sin (\omega t + \varphi^* + \Delta \phi), \] (5.49)

\[ \dot{\eta} = \frac{d}{\omega} \sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}} 2r^* \cos (\omega t + \varphi^* + \Delta \phi), \] (5.50)

\[ \xi = \frac{1}{\omega} 2r^* \sin (\omega t + \varphi^*), \] (5.51)

\[ \dot{\xi} = 2r^* \cos (\omega t + \varphi^*), \] (5.52)

where

\[ \Delta \phi = \arccos \left( \frac{1}{\sqrt{1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2}}} \right). \] (5.53)

Using the method of multiple scales, we can show that the System II.a experiences a supercritical Hopf bifurcation at \( \mu = 0 \). The periodic solution near the bifurcation (5.49)-(5.53) is equal\(^2\) to the solution (4.99)-(4.103) obtained

\(^2\)There exists a time shift of \( \pi \) between the solutions. Yet they are identical, since \( \Delta \varphi \) in (5.48) can be chosen accordingly.
5.2. Method of Multiple Scales applied to System II.a and System III.a

in Section 4.2.

The periodic solution for System III.a is found by simply setting $\theta = 1$ in (5.49)-(5.53), which yields

\begin{align}
\eta &= 2\sqrt{-\frac{\mu}{d (c_1 + 3c_2)}} \sin (t + \Delta \varphi), \quad (5.54) \\
\dot{\eta} &= 2\sqrt{-\frac{\mu}{d (c_1 + 3c_2)}} \cos (t + \Delta \varphi), \quad (5.55) \\
\xi &= 2\sqrt{-\frac{\mu}{d^3 (c_1 + 3c_2)}} \sin (t + \Delta \varphi), \quad (5.56) \\
\dot{\xi} &= 2\sqrt{-\frac{\mu}{d^3 (c_1 + 3c_2)}} \cos (t + \Delta \varphi). \quad (5.57)
\end{align}

We note that, besides the time shift of $\frac{\pi}{2}$, the solution (5.54)-(5.57) is equal to the solution (4.45)-(4.48) obtained in Section 4.1.
Chapter 6

Averaging

We have investigated the solutions near the Hopf bifurcation of System II.a and System III.a using the center manifold reduction in Chapter 4 and the method of multiple scales in Chapter 5. In this chapter we apply the method of averaging [16, 17] or also called the method of slowly changing phase and amplitude.

The method of averaging is a classical and useful computational technique for analyzing nonlinear problems with weak nonlinearities. The main idea of this method is to write the solution of the dynamical system as an oscillation with time varying phase and amplitude. Because the nonlinearities of the problem are assumed to be small, the phase and amplitude of the oscillation change slowly with time, which can be used for simplifying the analysis.

In Section 6.1 the method of averaging is applied to System III, i.e. with nonlinear Helmholtz damper. The analysis for the more general System II, i.e. also with nonlinear Helmholtz damper, is presented in Section 6.2.

6.1 Averaging of System III

The dynamics of System III is given by

\begin{align*}
\ddot{\eta} + (q + c_1\eta^2 + c_2\dot{\eta}^2)\dot{\eta} + \eta &= - (\mu - dq)\dot{\xi}, \\
\ddot{\xi} + |d + \delta\xi|\dot{\xi} + \xi &= \dot{\eta},
\end{align*}

(6.1) \hspace{1cm} (6.2)

where \(\mu = \epsilon + dq\) is chosen as the bifurcation parameter. We introduce new coordinates \(y\) and \(z\) as

\begin{align*}
y &= d\xi - \eta, \\
z &= q\xi + \eta.
\end{align*}

(6.3) \hspace{1cm} (6.4)
6.1. Averaging of System III

The inverse of the transformation (6.3)-(6.4) is given by

\[ \eta = \frac{y + z}{\nu}, \]
\[ \xi = \frac{dy - qz}{\nu}, \]

where

\[ \nu = d + q. \]

Rewriting the dynamics (6.1)-(6.2) in the new coordinates yields

\[ \ddot{y} + y = -E + F + dG + H, \]
\[ \ddot{z} + z = -F + qG - H, \]

where

\[ E = \nu \dot{y}, \]
\[ F = \mu \frac{\dot{y} + \dot{z}}{\nu}, \]
\[ G = \left( d - \frac{d + \delta y + z}{\nu} \right) \frac{\dot{y} + \dot{z}}{\nu}, \]
\[ H = \left( c_1 \frac{-qy + dz}{\nu} \right)^2 + c_2 \left( \frac{-q\dot{y} + d\dot{z}}{\nu} \right)^2 \frac{-q\dot{y} + d\dot{z}}{\nu}. \]

The solution to (6.8)-(6.13) for vanishing right-hand sides is given by

\[ y = A \cos t + B \sin t, \]
\[ z = C \cos t + D \sin t, \]

with the corresponding time derivative

\[ \dot{y} = -A \sin t + B \cos t, \]
\[ \dot{z} = -C \sin t + D \cos t. \]

For non-vanishing right-hand sides in (6.8)-(6.13), we assume that the solution of \( y \) and \( z \) can still be written in this form, but with time-varying coefficients \( A(t), B(t), C(t) \) and \( D(t) \). Therewith, it follows from (6.8)-(6.9) that

\[ -\dot{A} \sin t + \dot{B} \cos t = -E + F + dG + H, \]
\[ -\dot{C} \sin t + \dot{D} \cos t = -F + qG - H. \]
6.1. Averaging of System III

Furthermore, the derivative of (6.14) and (6.15) has to be equal to (6.16) and (6.17), respectively, that is

\[
\dot{A} \cos t + \dot{B} \sin t = 0, \quad (6.20)
\]
\[
\dot{C} \cos t + \dot{D} \sin t = 0. \quad (6.21)
\]

Solving (6.18)-(6.21) for \(\dot{A}(t), \dot{B}(t), \dot{C}(t)\) and \(\dot{D}(t)\) yields

\[
\dot{A} = E \sin t - F \sin t - dG \sin t - H \sin t, \quad (6.22)
\]
\[
\dot{B} = -E \cos t + F \cos t + dG \cos t + H \cos t, \quad (6.23)
\]
\[
\dot{C} = F \sin t - qG \sin t + H \sin t, \quad (6.24)
\]
\[
\dot{D} = -F \cos t + qG \cos t - H \cos t. \quad (6.25)
\]

Up to this point, we have only rewritten the original dynamics of System III in new coordinates without any simplifications. Therefore, the set of equations (6.22)-(6.25) fully describes the dynamics of System III.

Now, we assume that the amplitudes change slowly with time. Therefore, the right-hand sides of (6.22)-(6.25) must be small. Even though this may not be generally the case, this assumption is valid near the periodic solutions which we are interested in. According to the method of averaging, the terms which have a vanishing average over a period time \(T = 2\pi\) are omitted. We define two averaging operators as

\[
\bar{s}(\cdot) := \frac{1}{2\pi} \int_{0}^{2\pi} (\cdot) \sin t dt, \quad (6.26)
\]
\[
\bar{c}(\cdot) := \frac{1}{2\pi} \int_{0}^{2\pi} (\cdot) \cos t dt. \quad (6.27)
\]

Therefore, the averaged equations can be written as

\[
\dot{A} = \bar{s}E - \bar{s}F - d\bar{s}G - \bar{s}H, \quad (6.28)
\]
\[
\dot{B} = -\bar{c}E + \bar{c}F + d\bar{c}G + \bar{c}H, \quad (6.29)
\]
\[
\dot{C} = \bar{s}F - q\bar{s}G + \bar{s}H, \quad (6.30)
\]
\[
\dot{D} = -\bar{c}F + q\bar{c}G - \bar{c}H. \quad (6.31)
\]

Applying the operators (6.26) and (6.27) to the terms \(E, F, G\) and \(H\) yields
6.1. Averaging of System III

after several calculation steps

\[
\begin{align*}
\bar{E}^s &= -\frac{1}{2}\nu A, \quad (6.32) \\
\bar{E}^c &= \frac{1}{2}\nu B, \quad (6.33) \\
\bar{F}^s &= -\frac{1}{2}\mu (A + C), \quad (6.34) \\
\bar{F}^c &= \frac{1}{2}\mu (B + D), \quad (6.35) \\
\bar{G}^s &= \frac{1}{2}\nu (A + C)g(a), \quad (6.36) \\
\bar{G}^c &= -\frac{1}{2}\nu (B + D)g(a), \quad (6.37)
\end{align*}
\]

where

\[
g(a) = \begin{cases} 
\frac{2}{\pi} \left( \frac{\sqrt{a^2-1}(1+2a^2)}{3a^2} - \arccos \frac{1}{|a|} \right) & \text{for } |a| > 1, \\
0 & \text{for } |a| \leq 1,
\end{cases} \quad (6.38)
\]

\[
a = \frac{\delta}{\nu} \sqrt{(A + C)^2 + (B + D)^2}, \quad (6.39)
\]

and

\[
\begin{align*}
\bar{H}^s &= -\frac{c_1 + 3c_2}{8\nu^3} (-qA + dC) \left((qA + dC)^2 + (qB + dB)^2\right), \quad (6.40) \\
\bar{H}^c &= \frac{c_1 + 3c_2}{8\nu^3} (-qB + dD) \left((qA + dC)^2 + (qB + dB)^2\right). \quad (6.41)
\end{align*}
\]
6.1. Averaging of System III

Substituting (6.32)-(6.41) into (6.28)-(6.31) yields

\[ \dot{A} = -\frac{1}{2} \nu A + \frac{1}{2} \mu (A + C) - \frac{1}{2} \frac{d}{\nu}(A + C)g(a) + \frac{c_1 + 3c_2}{8\nu^3} (-qA + dC) ((-qA + dC)^2 + (-qB + dD)^2), \]  
(6.42)

\[ \dot{B} = -\frac{1}{2} \nu B + \frac{1}{2} \mu (B + D) - \frac{1}{2} \frac{d}{\nu}(B + D)g(a) + \frac{c_1 + 3c_2}{8\nu^3} (-qB + dD) ((-qA + dC)^2 + (-qB + dD)^2), \]  
(6.43)

\[ \dot{C} = -\frac{1}{2} \frac{\mu}{\nu}(A + C) - \frac{1}{2} \frac{d}{\nu}(A + C)g(a) - \frac{c_1 + 3c_2}{8\nu^3} (-qA + dC) ((-qA + dC)^2 + (-qB + dD)^2), \]  
(6.44)

\[ \dot{D} = -\frac{1}{2} \frac{\mu}{\nu}(B + D) - \frac{1}{2} \frac{d}{\nu}(B + D)g(a) - \frac{c_1 + 3c_2}{8\nu^3} (-qB + dD) ((-qA + dC)^2 + (-qB + dD)^2). \]  
(6.45)

The equilibria of the dynamical system (6.42)-(6.45) correspond to periodic solutions in \( y \)-\( z \)-coordinates and, according to the transformation (6.3)-(6.4), they correspond to periodic solutions in the original coordinates \( \eta \) and \( \xi \). We introduce new coordinates as

\[ K = A + C, \]  
(6.46)

\[ L = -qA + dC, \]  
(6.47)

\[ M = B + D, \]  
(6.48)

\[ N = -qB + dD. \]  
(6.49)

Rewriting (6.42)-(6.45) in the coordinates (6.46)-(6.49), the dynamics is obtained in a simpler form given by

\[ \dot{K} = -\frac{1}{2} d(1 + g(a))K + \frac{1}{2} L, \]  
(6.50)

\[ \dot{L} = \frac{1}{2} (dq - \mu)K - \frac{1}{2} \left( q + \frac{c_1 + 3c_2}{4\nu^2}(L^2 + N^2) \right) L, \]  
(6.51)

\[ \dot{M} = -\frac{1}{2} d(1 + g(a))M + \frac{1}{2} N, \]  
(6.52)

\[ \dot{N} = \frac{1}{2} (dq - \mu)M - \frac{1}{2} \left( q + \frac{c_1 + 3c_2}{4\nu^2}(L^2 + N^2) \right) N, \]  
(6.53)
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where

\[
g(a) = \begin{cases} 
\frac{2}{\pi} \left( \frac{\sqrt{a^2 - 1(1 + 2a^2)}}{3a^2} - \arccos \frac{1}{|a|} \right) & \text{for } |a| > 1, \\
0 & \text{for } |a| \leq 1,
\end{cases}
\]

(6.54)

\[
a = \delta d \nu \sqrt{K^2 + M^2}.
\]

(6.55)

Note that, according to the transformation (6.5)-(6.6) and the Ansatz (6.14)-(6.15), the amplitude of oscillation in the original coordinates \(\xi\) and \(\eta\) can be written as

\[
r_\xi = \frac{1}{\nu} \sqrt{(A + C)^2 + (B + D)^2} = \frac{1}{\nu} \sqrt{K^2 + M^2},
\]

(6.56)

\[
r_\eta = \frac{1}{\nu} \sqrt{(-qA + dC)^2 + (-qB + dD)^2} = \frac{1}{\nu} \sqrt{L^2 + N^2}.
\]

(6.57)

The eigenvalues corresponding to the trivial equilibrium are

\[
\lambda_{\{1,2,3,4\}} = \frac{1}{2} \left( \frac{-\nu}{2} \pm \sqrt{\left(\frac{\nu}{2}\right)^2 - \mu} \right).
\]

(6.58)

Therefore, the equilibrium at the origin is stable for \(\nu > 0 \land \mu > 0\), which corresponds to results (3.9)-(3.10) obtained from the eigenvalue analysis in Section 3.1.

The nontrivial equilibria of (6.50)-(6.53) are given by the implicit equations

\[
K^* + M^* = \frac{-4\nu^2(\mu + d g(a))}{d^2(c_1 + 3c_2)(1 + g(a))^2},
\]

(6.59)

\[
L^* + N^* = \frac{-4\nu^2(\mu + d g(a))}{d(c_1 + 3c_2)(1 + g(a))} = d^2(1 + g(a))^2(K^* + M^*).
\]

(6.60)

For a linear Helmholtz damper, i.e. \(\delta = 0\), we have \(a = 0\) and \(g(a) = 0\), according to (6.54) and (6.55). Therefore, according to (6.59)-(6.60), the equilibria can be written in a closed form as

\[
K^* + M^* = \frac{-4\nu^2\mu}{d^2(c_1 + 3c_2)},
\]

(6.61)

\[
L^* + N^* = \frac{-4\nu^2\mu}{d(c_1 + 3c_2)}.
\]

(6.62)
6.1. Averaging of System III

Substituting (6.61)-(6.62) into (6.56)-(6.57) yields

\[
\begin{align*}
    r_\xi &= \sqrt{-\frac{4\mu}{d^3(c_1 + 3c_2)}}, \\
    r_\eta &= \sqrt{-\frac{4\mu}{dc_1 + 3c_2}}.
\end{align*}
\]  

We note that for a linear Helmholtz damper a supercritical Hopf bifurcation occurs. The stable periodic solution in the original coordinates \(\eta\) and \(\xi\), which branches off at \(\mu = 0\), oscillates at a frequency of \(\omega = 1\) and the amplitude of oscillation is given by (6.63)-(6.64). This corresponds to the results (4.45)-(4.48) obtained by the center manifold reduction in Section 4.1 and the results (5.54)-(5.57) obtained from the method of multiple scales in Section 5.2.

To find the periodic solutions for a nonlinear Helmholtz damper, we rewrite the left-hand side of (6.59) using (6.55) and obtain

\[
a^2 - k_1^2k_2 + \frac{g(a)}{1 + g(a)} = 0,
\]

where

\[
\begin{align*}
    k_1 &= \frac{2\delta}{d^2}\sqrt{-\frac{q}{c_1 + 3c_2}}, \\
    k_2 &= \frac{\mu}{dq}.
\end{align*}
\]

Alternatively, (6.65) can be written as

\[
a^2 + \frac{k_3 - k_1^2g(a)}{(1 + g(a))^3} = 0,
\]

where

\[
k_3 = -k_1^2k_2 = \frac{4\delta^2\mu}{(c_1 + 3c_2)d^3}.
\]

We consider only negative \(q\), since for \(q > 0\) no periodic solutions exist, as it is shown in Section 3.2. According to the considered parameter range given by Table 2.1, the parameter \(k_1\) is always positive and \(k_2\) is positive for \(\mu < 0\) and negative for \(\mu > 0\). Therefore, the parameter \(k_3\) has the same sign as \(\mu\).

The roots of (6.68) correspond to equilibria of (6.50)-(6.53) and therefore to periodic solutions in the original coordinates \(\eta\) and \(\xi\), which oscillate at a frequency of \(\omega = 1\) and the amplitude of oscillation is given, according to
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(6.55)-(6.57) and (6.59)-(6.60), as

\[ r_\xi = \frac{1}{\nu} \sqrt{K^2 + M^2} = \frac{d}{\delta} a, \quad (6.70) \]

\[ r_\eta = \frac{1}{\nu} \sqrt{L^2 + N^2} = \frac{d^2}{\delta} (1 + g(a)) a. \quad (6.71) \]

The amplitude of the periodic solutions is found by solving the implicit equation (6.68) and substituting into (6.70)-(6.71). Figure 6.1 shows the amplitude of oscillation of \( \xi \), scaled by the factor \( \delta d \), as a function of \( k_1 \) for different values of \( k_3 \). The black lines correspond to stable periodic solutions, whereas the grey lines correspond to unstable periodic solutions. According to (6.54), we have

\[ g(a) = \frac{4}{3\pi} a \quad \text{for } a >> 1. \quad (6.72) \]

Substituting (6.72) into (6.68) yields

\[ a^2 = \frac{3\pi}{4} k_1 \quad \text{for } a >> 1. \quad (6.73) \]

Therefore, the radius of the stable periodic solution for high amplitude becomes

\[ r_\xi = \frac{d}{\delta} a = \frac{d}{\delta} \sqrt{\frac{3\pi}{4} k_1} = \sqrt{\frac{3\pi}{2\delta}} \left( \frac{q}{(c_1 + 3c_2)} \right)^{\frac{1}{4}}, \quad (6.74) \]

which is independent of \( \mu \). Therefore, the asymptotical behaviour of the stable periodic solutions whose amplitudes increase for an increasing \( k_1 \) in Figure 6.1 is given by (6.74).

The equilibrium is stable for \( k_3 \geq 0 \) and unstable otherwise. The first case, i.e. \( k_3 \geq 0 \), is depicted in Figure 6.2. For \( k_1 < 6.2 \) the equilibrium is stable and we have no coexisting periodic solutions. Nevertheless, this does not prove global asymptotic stability of the equilibrium, since we have not excluded quasiperiodic solutions or chaotic behaviour. The prove of global attractivity would require Lyapunov techniques. For \( k_1 > 6.2 \), depending on the parameter \( k_3 \), we have additionally a stable and an unstable periodic solution. Therefore, a fold bifurcation occurs under the influence of the parameter \( k_1 \) and the equilibrium is only locally asymptotically stable. Due to an external perturbation, the state of the system can escape the region of attraction of the equilibrium and enter the region of attraction of the coexisting periodic solution.

The second case, i.e. \( k_3 < 0 \), is depicted in Figure 6.3. For \( k_1 < 5.2 \) the unstable equilibrium is surrounded by one stable periodic solution. For \( k_1 > 5.2 \), depending on the parameter \( k_3 \), we have additionally a stable and
6.1. Averaging of System III

Figure 6.1: Stable (black) and unstable (grey) periodic solutions of System III for different values of $k_1$ and $k_3$. Depicted is the amplitude of oscillation of $\xi$, scaled by the factor $\frac{\delta}{\theta}$. The equilibrium is stable for $k_3 \geq 0$ and unstable for $k_3 < 0$. 
6.1. Averaging of System III

Figure 6.2: Stable (black) and unstable (grey) periodic solutions of System III for different values of \( k_1 \) and \( k_3 \geq 0 \). Depicted is the amplitude of oscillation of \( \xi \), scaled by the factor \( \frac{\delta}{d} \).
6.1. Averaging of System III

Figure 6.3: Stable (black) and unstable (grey) periodic solutions of System III for different values of $k_1$ and $k_3 < 0$. Depicted is the amplitude of oscillation of $\xi$, scaled by the factor $\frac{\delta}{d}$.

an unstable periodic solution and, thus, a fold bifurcation occurs under the influence of the parameter $k_1$. We have coexistence of two stable and two unstable limit sets. For $a = \frac{\delta}{d} r_\xi < 1$, equation (6.68) becomes

$$a^2 + k_3 = 0.$$  \hspace{1cm} (6.75)

Therefore, the radius of the unstable periodic solution with the smaller amplitude is obtained as

$$r_\xi = \frac{d}{\delta} a = \frac{d}{\delta} \sqrt{-k_3} = \sqrt{-\frac{4\mu}{d^2(c_1 + 3c_2)}},$$  \hspace{1cm} (6.76)

which is independent of $\delta$ and corresponds to (6.63). The periodic solution (6.76) exists only for $k_3 < 0$. They are shown in Figure 6.3 for $k_3 = -0.5$ and $k_3 = -1$, but they do not occur in Figure 6.2.

Considering (6.65) instead of (6.68), we can redraw Figure 6.1 in a more intuitive way. Figure 6.4 shows the scaled amplitude of oscillation of $\xi$ as a function of $k_2$ for different values of $k_1$. The black lines correspond to sta-
6.1. Averaging of System III

Figure 6.4: Stable (black) and unstable (grey) periodic solutions of System III for different values of $k_2$ and $k_1$. Depicted is the amplitude of oscillation of $\xi$, scaled by the factor $\frac{\delta}{d}$.

Table 6.1 shows the results of the averaging of System III. Stable periodic solutions, whereas the grey lines correspond to unstable periodic solutions. According to (6.67), the equilibrium is unstable for $k_2 > 0$ and stable otherwise. The parameter $k_1$, defined by (6.66), captures the linear and nonlinear damping in the system and, most importantly, it is proportional to $\delta$. Therefore, $k_1$ captures the nonlinearity of the Helmholtz damper. For a weakly nonlinear Helmholtz damper, i.e. $k_1 < 5.2$, the system experiences only a Hopf bifurcation under the influence of the parameter $k_2$. For $k_1 > 5.2$ two additional Fold bifurcations occur and we have coexistence of stable limit sets. Keeping $k_1$ fixed, e.g. at $k_1 = 6$, and slowly sweeping $k_2$ up and down, we follow the stable branches and ‘jump’ at the fold bifurcations from one periodic solution to the other, which leads to a hysteresis effect. For $k_1 > 6.2$ the ‘elbow’ in Figure 6.4 even reaches inside the region where $k_2 < 0$ and the equilibrium is only locally asymptotically stable.

At Section (1) (left dotted line in Figure 6.4), we have a stable equilibrium without coexisting periodic solutions for small $k_1$, which is depicted in Figure 6.5 labelled with (1a). For a highly nonlinear Helmholtz damper we have additionally a stable and an unstable periodic solution, which is shown in
6.1. Averaging of System III

Figure 6.5: The stable equilibrium either has no coexisting periodic solutions (left) or it is surrounded by a stable and an unstable periodic solution (right).

Figure 6.6: The unstable equilibrium is surrounded by either one stable periodic solution or it coexists with three periodic solutions with alternating stability (right).

Figure 6.5 labelled with (1b). At Section (2) (right dotted line in Figure 6.4), we have an unstable equilibrium, which is either surrounded by one stable periodic solution \((k_1 \text{ small})\), depicted in Figure 6.6 labelled with (2a), or it coexists with three periodic solutions with alternating stability \((k_1 \text{ large})\), which is shown in Figure 6.6 labelled with (2b).

The solid lines in Figure 6.7 shows at which values of \(k_1\) and \(k_2\) fold bifurcations occur. The two fold bifurcations meet at \((k_2, k_1) = (0.051, 5.2)\). At this point, the fold bifurcations cancel each other out and we have a cusp catastrophe, which is a bifurcation of codimension 2. One fold bifurcation tends to infinity for \(k_2 \to 0^+\). This can be seen be expanding (6.65) in a Taylor series around \(a = 1\) and omitting terms of higher order than \(O((a - 1)^2)\), which yields

\[
a^2 - k_1^2k_2 = 0. \tag{6.77}
\]

According to equation (6.77), the escaping fold bifurcation asymptotically ap-
6.1. Averaging of System III

Figure 6.7: System III experiences a Hopf bifurcation (dashed) and fold bifurcations (solid) under the influence of the parameters $k_1$ and $k_2$. A cusp catastrophe happens at $(k_2, k_1) = (0.051, 5.2)$.

The supercritical Hopf bifurcation occurs when crossing the dashed line at $k_2 = 0$ in Figure 6.7. The parameter space is partitioned into four regions (1a), (1b), (2a) and (2b). The periodic solutions occurring in each region are depicted in Figure 6.5 and Figure 6.6. For a weakly nonlinear Helmholtz resonator, i.e. $k_1 < 5.2$, we have only a Hopf bifurcation and no fold bifurcations. For $k_1 < 6.2$, the stable equilibrium has coexisting periodic solutions, as we have seen in Figure 6.2.

It might be impractical to design an experiment, where $k_2$ is changed continuously while keeping $k_1$ fixed or vice versa. Consider an experiment, where $q$ is changed continuously while keeping all other parameters constant. Therefore, we define the parameters

\[
\hat{k}_1 = \sqrt{-\frac{1}{q}}k_1 = \frac{2\delta}{d^2}\sqrt{-\frac{1}{c_1 + 3c_2}},
\]

\[
\hat{k}_2 = q(k_2 - 1) = \frac{\epsilon}{d},
\]
6.2 Averaging of System II

The dynamics of System II is given by

\[ \ddot{\eta} + (q + c_1 \eta^2 + c_2 \dot{\eta}^2) \dot{\eta} + \eta = -\epsilon \theta \dot{\xi}, \]  

(6.80)

\[ \ddot{\xi} + |d + \delta \xi| \dot{\xi} + \theta \xi = \dot{\eta}, \]  

(6.81)
6.2. Averaging of System II

Figure 6.9: Stable (black) and unstable (grey) periodic solutions of System III for $\hat{k}_2 = 0.02$ and different values of $q$ and $\hat{k}_1$. Depicted is the amplitude of oscillation of $\xi$, scaled by the factor $\frac{\delta}{d}$. The equilibrium is unstable for $q < -\hat{k}_2$. 

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6.2. Averaging of System II

where \( \mu = \epsilon \theta + dq \left( 1 + \frac{(\theta - 1)^2}{\nu_1 \nu_2} \right) \) is chosen as the bifurcation parameter. The parameters \( \nu_1 \) and \( \nu_2 \) are given by

\[
\begin{align*}
\nu_1 &= d + q, \\
\nu_2 &= d + q\theta,
\end{align*}
\]

and they are assumed to be positive. The same coordinates \( y \) and \( z \) are introduced as for System III, which are

\[
\begin{align*}
y &= d\xi - \eta, \\
z &= q\xi + \eta.
\end{align*}
\]

The coordinates \( \eta \) and \( \xi \) can be expressed in the new coordinates as

\[
\begin{align*}
\eta &= \frac{y + z}{\nu_1}, \\
\xi &= q\xi + \eta.
\end{align*}
\]

The dynamics (6.80)-(6.81) in the new coordinates \( y \) and \( z \) is obtained as

\[
\begin{align*}
\ddot{y} + y + d\frac{\theta - 1}{\nu_1} (y + z) + dq \frac{(\theta - 1)^2}{\nu_1^2 \nu_2} (\dot{y} + \dot{z}) &= -E + F + dG + H, \\
\ddot{z} + z + q\frac{\theta - 1}{\nu_1} (y + z) - dq \frac{(\theta - 1)^2}{\nu_1^2 \nu_2} (\dot{y} + \dot{z}) &= -F + qG - H,
\end{align*}
\]

where

\[
\begin{align*}
E &= \nu_1 \dot{y}, \\
F &= \mu \frac{\dot{y} + \dot{z}}{\nu_1}, \\
G &= \left( d - d + \frac{y + z}{\nu_1} \right) \frac{\dot{y} + \dot{z}}{\nu_1}, \\
H &= \left( c_1 \left( -qy + dz \right)^2 + c_2 \left( d\dot{z} - q\dot{y} \right)^2 \right) \frac{-q\dot{y} + d\dot{z}}{\nu_1}.
\end{align*}
\]

At \( \mu = 0 \), one pair of eigenvalues of the linearization of (6.80)-(6.81) cross the imaginary axis at \( \pm i\omega \), where \( \omega = \sqrt{\frac{\nu_2}{\nu_1}} \), as we have seen in Section 3.1. Hence,
6.2. Averaging of System II

we assume the solution to be of the form

\[ y(t) = A(t) \cos \omega t + B(t) \sin \omega t, \quad (6.94) \]
\[ \dot{y}(t) = -\omega A(t) \sin \omega t + \omega B(t) \cos \omega t, \quad (6.95) \]
\[ z(t) = C(t) \cos \omega t + D(t) \sin \omega t, \quad (6.96) \]
\[ \dot{z}(t) = -\omega C(t) \sin \omega t + \omega D(t) \cos \omega t. \quad (6.97) \]

Substituting the Ansatz (6.94)-(6.97) into the dynamics (6.88)-(6.89) yields

\[ -\dot{A} \omega \sin \omega t + \dot{B} \omega \cos \omega t - A(\omega^2 - 1) \cos \omega t - B(\omega^2 - 1) \sin \omega t \\
+ d\frac{\theta - 1}{\nu_1} ((A + C) \cos \omega t + (B + D) \sin \omega t) \\
+ dq\frac{(\theta - 1)^2}{\nu_1^2 \nu_2} ((B + D) \cos \omega t - (A + C) \sin \omega t) \omega \\
= -E + F + dG + H, \quad (6.98) \]
\[ -\dot{C} \omega \sin \omega t + \dot{D} \omega \cos \omega t - C(\omega^2 - 1) \cos \omega t - D(\omega^2 - 1) \sin \omega t \\
+ q\frac{\theta - 1}{\nu_1} ((A + C) \cos \omega t + (B + D) \sin \omega t) \\
- dq\frac{(\theta - 1)^2}{\nu_1^2 \nu_2} ((B + D) \cos \omega t - (A + C) \sin \omega t) \omega \\
= -F + qG - H. \quad (6.99) \]

Furthermore, the derivative of (6.94) and (6.96) has to be equal to (6.95) and (6.97), respectively, which yields

\[ \dot{A} \cos \omega t + \dot{B} \sin \omega t = 0, \quad (6.100) \]
\[ \dot{C} \cos \omega t + \dot{D} \sin \omega t = 0. \quad (6.101) \]

We introduce the same transformation as in Section 6.1 given by (6.46)-(6.49), that is

\[ K = A + C, \quad (6.102) \]
\[ L = -qA + dC, \quad (6.103) \]
\[ M = B + D, \quad (6.104) \]
\[ N = -qB + dD. \quad (6.105) \]
6.2. Averaging of System II

Rewriting (6.98)-(6.101) in the coordinates (6.102)-(6.105) and using the relation \( \omega^2 - 1 = q_2 \frac{\nu_1}{\nu_2} \) yields

\[
\begin{align*}
- \dot{K} \sin \omega t + \dot{M} \cos \omega t + d \frac{\theta - 1}{\nu_2} \omega (K \cos \omega t + M \sin \omega t) \\
= - \frac{1}{\omega} E + \frac{1}{\omega} \nu_1 G,
\end{align*}
\]

(6.106)

\[
\begin{align*}
- \dot{L} \sin \omega t + \dot{N} \cos \omega t - q \frac{\theta - 1}{\nu_2} \omega (L \cos \omega t + N \sin \omega t) \\
- dq \left( \frac{\nu_1}{\nu_2} \right)^2 (M \cos \omega t - K \sin \omega t) = \frac{1}{\omega} q E - \frac{1}{\omega} \nu_1 F - \frac{1}{\omega} \nu_1 H,
\end{align*}
\]

(6.107)

and

\[
\begin{align*}
\dot{K} \cos \omega t + \dot{M} \sin \omega t &= 0, \\
\dot{L} \cos \omega t + \dot{N} \sin \omega t &= 0.
\end{align*}
\]

(6.108) (6.109)

The system of equations (6.106)-(6.109) can be solved for \( \dot{K}(t) \), \( \dot{L}(t) \), \( \dot{M}(t) \) and \( \dot{N}(t) \) and we obtain

\[
\begin{align*}
\dot{K} &= d \frac{\theta - 1}{\nu_2} \omega (K \cos \omega t + M \sin \omega t) \sin \omega t \\
&\quad + \frac{1}{\omega} E \sin \omega t - \frac{1}{\omega} \nu_1 G \sin \omega t, \\
\dot{L} &= - q \frac{\theta - 1}{\nu_2} \omega (L \cos \omega t + N \sin \omega t) \sin \omega t \\
&\quad - dq \left( \frac{\nu_1}{\nu_2} \right)^2 (M \cos \omega t - K \sin \omega t) \sin \omega t \\
&\quad - \frac{1}{\omega} q E \sin \omega t + \frac{1}{\omega} \nu_1 F \sin \omega t + \frac{1}{\omega} \nu_1 H \sin \omega t, \\
\dot{M} &= - d \frac{\theta - 1}{\nu_2} \omega (K \cos \omega t + M \sin \omega t) \cos \omega t \\
&\quad - \frac{1}{\omega} E \cos \omega t + \frac{1}{\omega} \nu_1 G \cos \omega t, \\
\dot{N} &= q \frac{\theta - 1}{\nu_2} \omega (L \cos \omega t + N \sin \omega t) \cos \omega t \\
&\quad + dq \left( \frac{\nu_1}{\nu_2} \right)^2 (M \cos \omega t - K \sin \omega t) \cos \omega t \\
&\quad + \frac{1}{\omega} q E \cos \omega t - \frac{1}{\omega} \nu_1 F \cos \omega t - \frac{1}{\omega} \nu_1 H \cos \omega t.
\end{align*}
\]

(6.110) (6.111) (6.112) (6.113)

The dynamics (6.110)-(6.113) fully describes the dynamics of System II, since no simplifications have been made up to this point. Now, according to the
6.2. Averaging of System II

method of averaging, we assume that the amplitudes change slowly with time. Therefore, we omit terms with vanishing average over one period time $T = \frac{2\pi}{\omega}$. Using the same notation as in Section 6.1 defined by (6.26)-(6.27), the averaged equations are obtained as

$$
\dot{K} = \frac{d\theta}{2\nu_2} - \frac{1}{\nu_1} \frac{s}{\omega} M + \frac{1}{\omega} \frac{s}{\nu_1} E - \frac{1}{\nu_1} \frac{s}{\omega} \bar{G},
$$

(6.114)

$$
\dot{L} = -q\frac{\theta}{2\nu_2} N - dq\frac{(\theta - 1)^2}{2\nu_1\nu_2} K - \frac{1}{\omega} q \frac{s}{\nu_1} E + \frac{1}{\omega} \nu_1 \frac{s}{\omega} F + \frac{1}{\omega} \nu_1 \frac{s}{\omega} H,
$$

(6.115)

$$
\dot{M} = -d\frac{\theta}{2\nu_2} K - \frac{1}{\omega} \nu_1 \frac{c}{\omega} E + \frac{1}{\omega} \nu_1 \frac{c}{\omega} G,
$$

(6.116)

$$
\dot{N} = q\frac{\theta}{2\nu_2} L + dq\frac{(\theta - 1)^2}{2\nu_1\nu_2} M + \frac{1}{\omega} q \frac{c}{\nu_1} E - \frac{1}{\omega} \nu_1 \frac{c}{\omega} F - \frac{1}{\omega} \nu_1 \frac{c}{\omega} H,
$$

(6.117)

where

$$
\bar{E} = -\frac{1}{2} (dK - L) \omega,
$$

(6.118)

$$
\bar{E} = \frac{1}{2} (dM - N) \omega,
$$

(6.119)

$$
\bar{F} = -\frac{1}{2} \frac{\nu_1}{\nu_1} K \omega,
$$

(6.120)

$$
\bar{F} = \frac{1}{2} \frac{\mu}{\nu_1} M \omega,
$$

(6.121)

$$
\bar{G} = \frac{1}{2} \frac{d}{\nu_1} K g(a) \omega,
$$

(6.122)

$$
\bar{G} = -\frac{1}{2} \frac{d}{\nu_1} M g(a) \omega,
$$

(6.123)

$$
\bar{H} = -\frac{c_1 + 3c_2}{8\nu_1^3} L \left( L^2 + N^2 \right) \omega,
$$

(6.124)

$$
\bar{H} = \frac{c_1 + 3c_2}{8\nu_1^3} N \left( L^2 + N^2 \right) \omega,
$$

(6.125)

and

$$
g(a) = \begin{cases} 
\frac{2}{\pi} \left( \frac{\sqrt{a^2 - 1} + 2a^2}{3a^2} \right) & \text{for } |a| > 1, \\
0 & \text{for } |a| \leq 1,
\end{cases}
$$

(6.126)

$$
a = \frac{\delta}{dv_1} \sqrt{K^2 + M^2}.
$$

(6.127)
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The averaged equations are obtained by substituting (6.118)-(6.127) into (6.114)-(6.117) as

\[
\dot{K} = d\theta - \frac{1}{2} \nu^2 \omega M - \frac{1}{2} d(1 + g(a)) K + \frac{1}{2} L, \tag{6.128}
\]

\[
\dot{L} = -q\theta + \frac{1}{2} \nu^2 \omega N + dq \frac{(\theta - 1)^2}{2\nu_1 \nu_2} K
+ \frac{1}{2} (dq - \mu) K - \frac{1}{2} \left( q + \frac{c_1 + 3c_2}{\nu_1^2} (L^2 + N^2) \right) L, \tag{6.129}
\]

\[
\dot{M} = -d\theta - \frac{1}{2} \nu^2 \omega K - \frac{1}{2} d(1 + g(a)) M + \frac{1}{2} N, \tag{6.130}
\]

\[
\dot{N} = q\theta + \frac{1}{2} \nu^2 \omega L + dq \frac{(\theta - 1)^2}{2\nu_1 \nu_2} M
+ \frac{1}{2} (dq - \mu) M - \frac{1}{2} \left( q + \frac{c_1 + 3c_2}{4\nu_1^2} (L^2 + N^2) \right) N, \tag{6.131}
\]

where

\[
g(a) = \begin{cases} \frac{2}{3a^2} \left( \sqrt{a^2 - 1} + 2a^2 \right) & \text{for } |a| > 1, \\ 0 & \text{for } |a| \leq 1, \end{cases} \tag{6.132}
\]

\[
a = \frac{\delta}{d\nu_1} \sqrt{K^2 + M^2}. \tag{6.133}
\]

Note that the dynamics (6.128)-(6.131) for \( \theta = 1 \) corresponds to the dynamics (6.50)-(6.53) for System III obtained in Section 6.1. For \( \theta \neq 1 \), its has only an equilibrium at the origin. Other limit sets are found to be periodic solutions of the form

\[
K = \sqrt{X} \sin \omega, \tag{6.134}
\]

\[
L = \sqrt{X} \cos \omega, \tag{6.135}
\]

\[
M = \sqrt{Y} \sin (\omega + \Delta \phi), \tag{6.136}
\]

\[
N = \sqrt{Y} \cos (\omega + \Delta \phi). \tag{6.137}
\]

Thus, according to the Ansatz (6.94)-(6.97) and the transformations (6.86)-(6.87) and (6.102)-(6.105), the dynamics in the original coordinates \( \eta \) and \( \xi \) is
6.2. Averaging of System II

obtained as

\[
\eta = \frac{1}{\nu_1} \sqrt{V} \left( \sin(\tilde{\omega}t + \Delta \phi) \cos \omega t + \cos(\tilde{\omega}t + \Delta \phi) \sin \omega t \right) \\
= \frac{1}{\nu_1} \sqrt{V} \sin((\omega + \tilde{\omega})t + \Delta \phi), \tag{6.138}
\]

\[
\xi = \frac{1}{\nu_1} \sqrt{X} \left( \sin \tilde{\omega}t \cos \omega t + \cos \tilde{\omega}t \sin \omega t \right) \\
= \frac{1}{\nu_1} \sqrt{X} \sin(\omega + \tilde{\omega})t. \tag{6.139}
\]

For \( \mu = 0 \), the frequency \( \tilde{\omega} \) has to vanish and the original states \( \eta \) and \( \xi \) oscillate with the frequency \( \omega \). For \( \mu \neq 0 \), the frequency of oscillation is perturbed by \( \tilde{\omega} \), which corresponds to the solution (4.99)-(4.102) obtained by the center manifold reduction in Section 4.2 or the solution (5.49)-(5.52) obtained by the method of multiple scales in Section 5.2.

The amplitudes \( \sqrt{X} \), \( \sqrt{Y} \) and the phase shift \( \Delta \phi \) in (6.138)-(6.139) are not a function of time. Therefore, we define new coordinates as

\[
X = K^2 + M^2, \tag{6.140}
\]

\[
Y = N^2 + L^2, \tag{6.141}
\]

\[
Z = LK + MN \left( = \sqrt{XY} \cos \Delta \phi \right), \tag{6.142}
\]

\[
W = ML - KN \left( = \sqrt{XY} \sin \Delta \phi \right). \tag{6.143}
\]

We rewrite the dynamics (6.128)-(6.131) in the coordinates (6.140)-(6.143) and obtain

\[
\dot{X} = Z - d(1 + g(a))X, \tag{6.144}
\]

\[
\dot{Y} = \frac{dq(\theta - 1)^2}{\nu_1\nu_2} Z + \left( dq - \mu \right) Z - \left( q + \frac{c_1 + 3c_2}{4\nu_1^2} Y \right) Y, \tag{6.145}
\]

\[
2\dot{Z} = \frac{(\theta - 1)}{\omega} W + \frac{dq(\theta - 1)^2}{\nu_1\nu_2} X + \left( dq - \mu \right) X \\
- \left( q + \frac{c_1 + 3c_2}{4\nu_1^2} Y \right) Z + Y - d(1 + g(a))Z, \tag{6.146}
\]

\[
2\dot{W} = -\frac{(\theta - 1)}{\omega} Z - d(1 + g(a))W - \left( q + \frac{c_1 + 3c_2}{4\nu_1^2} Y \right)W. \tag{6.147}
\]

Besides the equilibrium at the origin, the dynamics (6.144)-(6.147) has non-trivial equilibria. Yet, the equilibria have not been found in closed form. Additionally, no relation between the parameters and the number of equilibria could have been found.
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Simulations of the dynamics (6.144)-(6.147) show a similar behaviour as we have seen for System III in Section 6.1. The trivial equilibrium is unstable for $\mu < 0$ and stable otherwise. Next to the unstable equilibrium at the origin, there exists a stable nontrivial equilibrium. Independently of the stability of the trivial equilibrium, increasing the nonlinearity of the Helmholtz resonator causes a saddle-node bifurcation and two new equilibria are created of which one is stable and the other is unstable.
Chapter 7
Conclusions

The equilibrium of the perfectly tuned model is globally asymptotically stable if the linear damping of the pressure oscillation in the combustion chamber is positive. For a negative linear damping, i.e. thermoacoustic instability occurs in the combustion chamber and the use of a Helmholtz damper is necessary, the equilibrium becomes unstable for $\mu = \epsilon + dq < 0$. This can occur for example if the dimension of the Helmholtz damper is too small compared to the combustion chamber. At this point, one pair of complex conjugate eigenvalues cross the imaginary axis and a supercritical Hopf bifurcation occurs. The stable periodic solutions branching off at this point are approximated using three different methods, which are the center manifold reduction, the method of multiple scales and the method of averaging. For a weakly nonlinear damping in the Helmholtz resonator, no other limit sets have been found. Yet, for a Helmholtz resonator with highly nonlinear damping, a fold bifurcation occurs and therefore two additional periodic solutions are created of which one is stable and the other is unstable. For this case, the stable equilibrium is only locally attractive. If the equilibrium is unstable, it is surrounded by three periodic solutions with alternating stability.

The stability analysis for the model with static detuning shows a similar behaviour as for the perfectly tuned model. For positive linear damping of the combustor dynamics, the equilibrium is globally asymptotically stable and for negative damping the equilibrium becomes unstable for $\mu = \epsilon \theta + dq \left(1 + \frac{(\theta - 1)^2}{(d + q)(d + q\theta)}\right) < 0$. When the equilibrium becomes unstable, a stable periodic solution branches off due to the nonlinear damping of the combustor dynamics and a supercritical Hopf bifurcation occurs. The pressure inside the combustion chamber and the acoustic velocity in the neck of the Helmholtz damper oscillate at the same frequency. Yet, which is in contrast to the perfectly tuned model, there exists a phase shift between the oscillations of the two subsystems and the frequency of oscillation depends on the bifurcation parameter.
The method of averaging is applied to the dynamics including the nonlinearity of the Helmholtz resonator. Yet, the equilibria of the averaged equations could not have been found in closed form in order to create a bifurcation diagram like for the perfectly tuned case. Fold bifurcations, as they occur in the perfectly tuned model, are expected due to simulations, yet these bifurcations could not have been shown analytically. Additional bifurcations for the model with static detuning are not excluded. One possibility to continue this investigation is the use of numerical methods such as numerical continuation.

Several aspects are not treated in this thesis which could be part of further investigations. A model including dynamical detuning is derived at the beginning of this thesis. Even though dynamical detuning is relevant in industry, this model could not have been investigated within the scope of this thesis. Furthermore, for certain choices of the system parameters, the equilibrium of the perfectly tuned model can become unstable even for $\mu > 0$. For this case, all four eigenvalues cross the imaginary axis simultaneously and therefore the dynamics can not be simplified using a center manifold reduction. This case has not been investigated and the bifurcation occurring at this point would be of interest at least from a mathematical point of view.
Appendix A

Method of Normal Forms

The normal form of a bifurcation is the simplest equation representing the type of bifurcation. In this section it is shown how to compute the normal form of the equation

\[ \ddot{u} + \omega^2 u = \kappa^2 f(u, \dot{u}) = \kappa^2 \left( \mu_1 u + \mu_2 \dot{u} + a_1 u^3 + a_2 u^2 \ddot{u} + a_3 u \dot{u}^2 + a_4 \dot{u}^3 \right). \]  

(A.1)

The undamped harmonic oscillator \( \ddot{u} + \omega^2 u = 0 \) is weak, which is captured by the small nondimensional parameter \( \kappa^2 \). Assuming that \( \omega^2 > \kappa^2 \mu_1 \), the equilibrium is stable for \( \mu_2 < 0 \) and unstable for \( \mu_2 > 0 \). The parameter \( \mu_2 \) is the bifurcation parameter and a Hopf bifurcation occurs at \( \mu_2 = 0 \). The amplitude of the periodic solution near the bifurcation depend on the parameters \( a_1-a_4 \) and \( \mu_2 \). The derivation for \( \mu_1 = 0 \) can be found in more detail in [6].

The only term of the perturbation \( f(u, \dot{u}) \), which is linear and independent of the bifurcation parameter \( \mu_2 \), is the term \( \mu_1 u \). Thus, we rewrite equation (A.1) as

\[ \ddot{u} + \hat{\omega}^2 u = \kappa^2 \hat{f}(u, \dot{u}) = \kappa^2 \left( \mu_2 \dot{u} + a_1 u^3 + a_2 u^2 \ddot{u} + a_3 u \dot{u}^2 + a_4 \dot{u}^3 \right), \]  

(A.2)

where

\[ \hat{\omega} = \sqrt{\omega^2 - \kappa^2 \mu_1}. \]  

(A.3)

We introduce new variables

\[ \zeta := \frac{1}{2} \left( u - \frac{i}{\hat{\omega}} \dot{u} \right), \]  

(A.4)

\[ \tilde{\zeta} := \frac{1}{2} \left( u + \frac{i}{\hat{\omega}} \dot{u} \right), \]  

(A.5)
where $\bar{\zeta}$ is the complex conjugate of $\zeta$. The state $u$ and its time derivative can be expressed in the new variables as

\begin{align}
  u &:= \zeta + \bar{\zeta}, \quad (A.6) \\
  \dot{u} &:= i\hat{\omega}(\zeta - \bar{\zeta}). \quad (A.7)
\end{align}

Differentiating (A.4) w.r.t. time and substituting (A.1) yields

\begin{equation}
  \dot{\zeta} := i\hat{\omega} \frac{1}{2} \left( u - \frac{i}{\hat{\omega}} \dot{u} \right) - \frac{i}{2\hat{\omega}} \kappa^2 f(u, \dot{u}). \quad (A.8)
\end{equation}

Using (A.4), (A.6) and (A.7), equation (A.8) becomes

\begin{align*}
  \dot{\zeta} &= i\hat{\omega} \zeta - \frac{i}{2\hat{\omega}} \kappa^2 f(\zeta + \bar{\zeta}, i\hat{\omega}(\zeta - \bar{\zeta})) \\
  &= i\hat{\omega} \zeta - \frac{i}{2\hat{\omega}} \kappa^2 \left( \mu_1(\zeta + \bar{\zeta}) + i\hat{\omega} \mu_2(\zeta - \bar{\zeta}) + a_1(\zeta + \bar{\zeta})^3 + i\hat{\omega} a_2(\zeta + \bar{\zeta})^2(\zeta - \bar{\zeta}) \\
  &\quad - \hat{\omega}^2 a_3(\zeta + \bar{\zeta})(\zeta - \bar{\zeta})^2 - i\hat{\omega}^3 a_4(\zeta - \bar{\zeta})^3 \right). \quad (A.9)
\end{align*}

In order to eliminate the non-resonant terms, we define a near-identity transformation as

\begin{equation}
  \zeta = \eta + \kappa^2 h(\eta, \bar{\eta}). \quad (A.10)
\end{equation}

Note that the order of perturbation in (A.10) is the same as the order of perturbation of the original equation (A.1). Substituting the transformation into (A.9) yields

\begin{equation}
  \dot{\eta} = i\hat{\omega}(\eta + \kappa^2 h) - \kappa^2 (h\dot{\eta} + h\dot{\bar{\eta}}) \\
  - \frac{i}{2\hat{\omega}} \kappa^2 f \left( \eta + \bar{\eta} + \kappa^2 (h + \bar{h}), i\hat{\omega} \left( \eta - \bar{\eta} + \kappa^2 (h - \bar{h}) \right) \right). \quad (A.11)
\end{equation}

Since the perturbation $f(\eta, \dot{\eta})$ contains only linear and third-order terms, $h$ is chosen as

\begin{equation}
  h(\eta, \bar{\eta}) = \Delta_1 \eta + \Delta_2 \bar{\eta} + \Lambda_1 \eta^3 + \Lambda_2 \eta^2 \bar{\eta} + \Lambda_3 \eta \bar{\eta}^2 + \Lambda_4 \eta^3. \quad (A.12)
\end{equation}

The terms $\dot{\eta}$ and $\dot{\bar{\eta}}$ in the right-hand side of (A.11) are found by iteration. To the first approximation, they can be replaced by $\dot{\eta} = i\hat{\omega} \eta$ and $\dot{\bar{\eta}} = -i\hat{\omega} \bar{\eta}$. 

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Substituting (A.12) into (A.11) and keeping terms up to Order $O(\kappa^2)$ yields

$$\dot{\eta} = i \left( \omega - \frac{\mu_1}{2\omega} \kappa^2 \right) \eta + \frac{1}{2} \kappa^2 \mu_2 \eta + 2i\omega \kappa^2 \left( \Delta_2 + \frac{i\mu_2}{4\omega} \right) \bar{\eta}$$

$$+ i\omega \kappa^2 \left( -2\Lambda_1 - \frac{1}{2\omega^2} (a_1 + i\omega a_2 - \omega^2 a_3 - i\omega^3 a_4) \right) \eta^3$$

$$+ i\omega \kappa^2 \left( -\frac{1}{2\omega^2} (3a_1 + i\omega a_2 + \omega^2 a_3 + 3i\omega^3 a_4) \right) \eta^2 \bar{\eta}$$

$$+ i\omega \kappa^2 \left( 2\Lambda_3 - \frac{1}{2\omega^2} (3a_1 - i\omega a_2 + \omega^2 a_3 - 3i\omega^3 a_4) \right) \eta^2 \bar{\eta}$$

$$+ i\omega \kappa^2 \left( 4\Lambda_4 - \frac{1}{2\omega^2} (a_1 - i\omega a_2 - \omega^2 a_3 + i\omega^3 a_4) \right) \bar{\eta}^3. \quad (A.13)$$

The parameters $\Delta_1$ and $\Lambda_2$ are arbitrary since they do not appear in (A.13). Therefore, the terms $\eta$ and $\eta^2 \bar{\eta}$ are called resonant terms because they can not be eliminated. The parameters $\Delta_2$, $\Lambda_1$, $\Lambda_3$ and $\Lambda_4$ can be chosen to eliminate the non-resonant terms $\bar{\eta}$, $\eta^3$, $\eta \bar{\eta}^2$ and $\bar{\eta}^3$; that is

$$\Delta_2 = -\frac{i\mu_2}{4\omega}, \quad (A.14)$$

$$\Lambda_1 = -\frac{1}{4\omega^2} (a_1 + \omega a_2 - \omega^2 a_3 - i\omega^3 a_4), \quad (A.15)$$

$$\Lambda_3 = -\frac{1}{4\omega^2} (3a_1 - i\omega a_2 + \omega^2 a_3 - 3i\omega^3 a_4), \quad (A.16)$$

$$\Lambda_4 = -\frac{1}{8\omega^2} (a_1 - i\omega a_2 - \omega^2 a_3 + i\omega^3 a_4). \quad (A.17)$$

Substituting (A.14)-(A.17) into (A.13), the dynamics of $\eta$ reduces to

$$\dot{\eta} = i \left( \omega - \frac{\mu_1}{2\omega} \kappa^2 \right) \eta + \frac{1}{2} \kappa^2 \mu_2 \eta - \frac{i}{2\omega} \kappa^2 \left( 3a_1 + i\omega a_2 + \omega^2 a_3 + 3i\omega^3 a_4 \right) \eta^2 \bar{\eta}. \quad (A.18)$$

Writing $\eta$ in polar coordinates, i.e. $\eta = re^{i\varphi}$, and separating real and imaginary parts, we obtain

Re: $\dot{r} = \frac{1}{2} \kappa^2 \mu_2 r + \frac{1}{2} \kappa^2 (a_2 + 3\omega^2 a_4) r^3, \quad (A.19)$

Im: $\dot{\varphi} = \omega - \frac{1}{2\omega} \kappa^2 \mu_1 - \frac{1}{2\omega} \kappa^2 (3a_1 + \omega^2 a_3) r^2. \quad (A.20)$

We note that the radius $r$ experiences a pitchfork bifurcation at $\mu_2 = 0$ and, thus, $\eta$ experiences a Hopf bifurcation at $\mu_2 = 0$. For $a_2 + 3\omega^2 a_4 > 0$ it

Note that the Taylor series expansion of (A.3) is given as $\hat{\omega} = \omega - \frac{\mu_1}{2\omega} \kappa^2 + O(\kappa^4)$.
is a supercritical pitchfork/Hopf bifurcation and for \( a_1 + 3\omega^2a_2 < 0 \) it is a subcritical pitchfork/Hopf bifurcation. The nontrivial equilibrium of (A.19) and (A.20) is

\[
\begin{align*}
    r^* &= \sqrt{-\frac{\mu_2}{a_2 + 3\omega^2a_4}}, \\
    \varphi^* &= \omega \left( 1 - \frac{k^2}{2\omega^2} \left( \mu_1 - \frac{3a_1 + \omega^2a_3}{a_2 + 3\omega^2a_4} \mu_2 \right) \right) t + \Delta \varphi,
\end{align*}
\]

which represents a harmonic oscillation of \( \eta \). According to the transformations (A.4), (A.5) and (A.10), we have \( u = \zeta + \bar{\zeta} \approx \eta + \bar{\eta} \). Therefore, the periodic solution of \( u \) is obtained as

\[
    u = r^* e^{i\varphi^*} + r^* e^{-i\varphi^*} = 2r^* \cos \varphi^*,
\]

where \( r^* \) and \( \varphi^* \) are given by (A.21) and (A.22).
Appendix B

Condition for vanishing secular terms

Consider a differential equation of the form
\[ \dot{x} - Ax = fe^{\lambda k t}, \]  
(B.1)
where the matrix \( A \in \mathbb{R}^n \) has an eigenvalue at \( \lambda_k \). We want to show that the solution \( x \) does not contain secular term if the vector \( f \) on the right-hand side of (B.1) is perpendicular to the left eigenvector \( q_k \) of the matrix \( A \) corresponding to the eigenvalue \( \lambda_k \).

We define the left eigenvectors \( q_i \) and the right eigenvectors \( p_i \) of \( A \) according to the eigenvalues \( \lambda_i \) as
\[ Ap_i = \lambda_i p_i \quad \forall i \in \{1, 2, \ldots, n\}, \]  
(B.2)
\[ A^T q_i = \lambda_i q_i \quad \forall i \in \{1, 2, \ldots, n\}. \]  
(B.3)
Furthermore, we define the matrices \( P = \{p_i\} \) and \( Q = \{q_i\} \) whose column vectors are the eigenvectors \( p_i \) and \( q_i \), respectively. Therefore, we can write the matrix \( A \) and its transpose as
\[ A = P \Lambda P^{-1}, \]  
(B.4)
\[ A^T = Q \Lambda Q^{-1}, \]  
(B.5)
where the Jordan Matrix \( \Lambda \) is
\[ \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix}. \]  
(B.6)
We define a new state $\zeta$ as

$$x = P\zeta. \quad (B.7)$$

Substituting (B.7) and (B.4) into (B.1) and using that the matrix $Q^T$ is the inverse of $P$ yields

$$\dot{\zeta} - \Lambda \zeta = Q^T f e^{\lambda_k t}. \quad (B.8)$$

The right-hand side of (B.8) can be written as

$$Q^T f = \sum_i (q_i^T f) e_i. \quad (B.9)$$

We choose the Ansatz for $\zeta$ as

$$\zeta = \sum_i a_i e_i e^{\lambda_i t} + \sum_{i \neq k} b_i e_i e^{\lambda_k t} + c e_k e^{\lambda_k t}, \quad (B.10)$$

where the third term, which is proportional to $c$, is a secular term. Using this Ansatz and (B.9), the differential equation (B.8) becomes

$$\sum_{i \neq k} b_i e_i \lambda_k e^{\lambda_k t} - \sum_{i \neq k} b_i e_i \lambda_i e^{\lambda_k t} + ce_k e^{\lambda_k t} = \sum_i (q_i^T f) e_i e^{\lambda_k t}. \quad (B.11)$$

We project the vector valued equation (B.11) into the direction of $e_i$ and obtain

$$b_i (\lambda_k - \lambda_i) = q_i^T f \quad \text{for } i \neq k, \quad (B.12)$$

$$c = q_k^T f \quad \text{for } i = k. \quad (B.13)$$

It follows from (B.10) and (B.13) that $\zeta$ does not contain secular terms if $q_k^T f$. Therefore, $x$ does not contain secular terms, if the vector $f$ is perpendicular to the left eigenvector $q_k$ of $A$ corresponding to the eigenvalue $\lambda_k$. 

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Bibliography


Selbstständigkeitserklärung

Ich erkläre mit meiner Unterschrift, das Merkblatt Plagiat zur Kenntnis genommen, die vorliegende Arbeit selbständig verfasst und die im betroffenen Fachgebiet üblichen Zitiervorschriften eingehalten zu haben. 
Merkblatt Plagiat: 
http://www.ethz.ch/students/semester/plagiarism_s_de.pdf

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