Trajectory Optimization of Mechanical Hybrid Systems Using SUMT

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Abstract— **The aim of this report is to propose a unified framework for the determination of non-smooth trajectories for structure-variant mechanical systems along with a computational scheme. The benefits to represent the dynamics as a measure-differential inclusion will be presented. The optimal control problem will be transcribed into a Nonlinear Programming Problem (NLP) and transformed from the infinite dimensional representation into a finite dimensional representation. The relation to bilevel programming will be established. A numerical scheme will be proposed for the determination of the state and costate trajectories, which can bear discontinuities and set-valuedness.**

I. INTRODUCTION

The trajectory determination of structure-variant systems which are exposed to high degree of nonlinearity, discontinuities in states and change of degrees of freedom (DOF) is a very challenging active research area as stated in [9], [10], [16]. The trajectory optimization of structure-variant mechanical systems belong to the class of mathematical programs with equilibrium constraints (MPEC). In [7] a MPEC is defined as an optimization problem in which the essential constraints are defined by parametric variational inequality or complementarity systems. One of the many representations of a MPEC can be stated as follows:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{z}} \quad & f(\mathbf{x},\mathbf{z}) \\ \mathbf{z} \;\in\; & \mathcal{S}(\mathbf{x}) \\ \mathbf{x} \;\in\; & \mathcal{U}_{ad},\, \mathbf{z} \in \mathcal{Z}. \end{aligned} \tag{1}$$

The problem stated in (1) includes a subclass of so-called bilevel programs, where $\mathcal{S}$ assigns each $\mathbf{x} \in \mathcal{U}_{ad}$ the solution of a "lower-level" optimization problem. In the case where the complementarity system arises from mechanical systems with unilateral contacts without Coulomb type of friction, a so-called subclass of MPEC, namely, bilevel programs apply. In references [11] and [12] a detailed treatment of complementarities and optimization can be found. On the other hand, systems with discontinuities in the states and non-differentiability in the classical sense have lead to the development of mathematical tools such as convex analysis [3], subdifferential calculus and non-smooth analysis. The outcome of these efforts is the representation of non-smooth mechanical systems by measure-differential inclusions (MDI) deeply rooted in the seminal works [14], [5] of J. J. Moreau.

The seminal works of J.J. Moreau [5] and [14] enabled a sound description of non-smooth mechanical systems as MDI and several good reference books are given in [1], [2] that have been published recently. In the framework of MPEC, the measure-differential inclusion (MDI) that describes the dynamics can be considered as the necessary condition of a "lower-level" optimization problem represented by the saddle-region restraining set $\mathcal{S}$. Here the time or energy could be the goal function to be minimized and the control action is represented by $\mathbf{x} \in \mathcal{U}_{ad}$. The controls can be considered as the variables of the "higher-level" optimization problem whereas the contact forces and mechanical states are variables of the "lower-level" (quasi-)optimization problem. In this report an augmented Lagrangian based sequential unconstrained minimization technique (SUMT) is used in order to solve the Nonlinear Mathematical Programming Problem (NLP). The time-stepping approach will be considered and the gradient-based minimization of the augmented Lagrangian will be used as a method to obtain the optimum. References [7] and [8] treat MPEC and bilevel programs extensively.

II. THE STRUCTURE-VARIANT MECHANICAL OPTIMAL CONTROL PROBLEM

The infinite dimensional general optimal control problem that is being investigated will be stated in this section. A very good reference on functions with bounded variations in time is [5].

Since the optimal control formulation is supposed to entail impulsive control actions as well as discontinuities in the generalized velocities, the velocities are considered as being functions of bounded variations (BV) allowing a countable number of finite discontinuities. Instead of assuming differentiability of $\mathbf{x}$, the states $\mathbf{x}$ will be assumed to be absolutely continuous everywhere on $t \in I = [t_0, t_f]$ and differentiable except at finite number of times denoted by the set $\{t_i\}$. A generalized velocity function $\mathbf{u} : I \to \mathcal{R}^n, \mathbf{u} \in \mathrm{BV}(I, \mathcal{R}^n)$ is defined so that absolutely continuous displacements $\mathbf{x}$ are obtained. The velocities $\dot{\mathbf{x}}$ of the system become the continuous part $\mathbf{u}_c$ of

$\mathbf{u}$ and the diffuse measure of du is not considered,

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{u}(\sigma) d\sigma. \tag{2}$$

The discontinuities in the generalized velocities can emanate from impulsive control actions as well as intrinsic system behaviour that might generate impacts. It is worth to elucidate the nature of the differential measures $d\mathbf{\Lambda}$ at this place. The differential measures can be a part of the control strategy which causes discontinuities in $\mathbf{u}$ or can be induced by the intrinsic system dynamics. The differential measures $d\mathbf{\Lambda}$ can be thought of consisting like in the decomposition of $\mathbf{u} \in \mathrm{BV}(I, \mathrm{R}^n)$ of two parts,

$$d\mathbf{\Lambda} = \mathbf{\Lambda}\, dt + \mathbf{\Delta}\, d\eta, \tag{3}$$

where $\mathbf{\Lambda}$ may consist of Lebesgue-measurable forces/controls and $\mathbf{\Delta}$ is regarded as a purely atomic impact impulsion. Controls which have only absolutely continuous character will be separated in the notation from the differential measures.

The non-smooth optimal control problem subject to a mechanical dynamical system described as a measure-differential inclusion can be stated as follows:

$$\begin{aligned}
&\min\, \mathcal{J}(\mathbf{x},\, \mathbf{u},\, \boldsymbol{\tau},\, d\mathbf{\Lambda},\, t) \\
&= \Phi(\mathbf{x}(t_f)\,, t_f) + \int_{t_0}^{t_f} g(\mathbf{u},\, \mathbf{x},\, \boldsymbol{\tau},\, d\mathbf{\Lambda})\, dt, && (4)\\
&d\mathbf{u} = \mathbf{f}(\mathbf{u},\, \mathbf{x},\, \boldsymbol{\tau},\, t)\, dt + \mathbf{p}(\mathbf{x},\, t)\; d\mathbf{\Lambda}, && (5)\\
&d\mathbf{\Lambda} \in \mathbf{\Upsilon}(\, d\mathbf{\Lambda},\, \mathbf{u},\, \mathbf{x}, \boldsymbol{\tau}), && (6)\\
&\mathbf{\Pi}(\mathbf{u},\, \mathbf{x},\, \boldsymbol{\tau}) \leq \mathbf{0}, && (7)\\
&\mathbf{\Psi}(\mathbf{x}(t_0),\, \mathbf{u}(t_0),\, \mathbf{x}(t_f),\, \mathbf{u}(t_f)) = \mathbf{0}, && (8)\\
&t_0 \quad \text{fixed},\, t_f \quad \text{free}, \quad t \in [t_0, t_f], && (9)
\end{aligned}$$

with absolutely continuous state variables $\mathbf{x} \in \mathrm{R}^n$, BV generalized velocities $\mathbf{u} \in \mathrm{R}^n$, control variables $\boldsymbol{\tau} \in \mathrm{R}^m$, measure differential forces and controls $d\mathbf{\Lambda} \in \mathrm{R}^p$, smooth system dynamics $\mathbf{f} : \mathrm{R}^n \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R} \rightarrow \mathrm{R}^n$, set-valued constraints on the differential measures $\mathbf{\Upsilon} : \mathrm{R}^p \times R^n \times R^n \times \mathrm{R}^m \rightarrow \mathrm{R}^p$, influence matrix of differential measures $\mathbf{p} : \mathrm{R} \rightarrow \mathrm{R}^n$, C^0 state and control constraints $\mathbf{\Pi} : \mathrm{R}^n \times \mathrm{R}^m \rightarrow \mathrm{R}^p$ and boundary constraints $\mathbf{\Psi} : \mathrm{R}^n \times \mathrm{R}^n \times \mathrm{R}^n \times \mathrm{R}^n \rightarrow \mathrm{R}^q$, C^0 end state cost $\Phi : \mathrm{R}^n \times R \rightarrow \mathrm{R}$, integrand of the cost functional $g : \mathrm{R}^n \times \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^p \rightarrow \mathrm{R}$. The cost functional $\mathcal{J}$ is assumed to be absolutely continuous since the integrand g is a function of bounded variation. The bounded variations of g emanates from the fact that the integrand can also depend on the generalized velocities $\mathbf{u} : I \rightarrow \mathrm{R}^n, \mathbf{u} \in \mathrm{BV}(I, \mathrm{R}^n)$. The set-valued dynamics depend on differential measures via the impulsive influence matrix $\mathbf{p}$.

The systems described by MDI can change degrees of freedom (.e.g stick-slip transitions) and exhibit discontinuities in velocities (.e.g. due to impacts). They are representative to structure-variant systems with explicit or implicit phase transitions. The difference between implicit and explicit phase transition is that the next phase is initiated by a controller strategy in the explicit case whereas in the implicit case the transition is invoked by the intrinsic system dynamics as in the case of a block that undergoes a stick-slip transition due to the frictional contact. These phase transitions may be accompanied by discontinuities in the states. The above given formulation entails trajectory optimization problems with unilateral contacts on position level and/or friction of Coulomb type.

III. THE NUMERICAL OPTIMIZATION SCHEME BASED ON SUMT

The basic idea in penalty methods is to penalize some or all of the constraints and add to the cost function a penalty term that adds a high cost for infeasibility. The penalty parameter c determines the amount of the penalty and is a measure to what extend the resulting unconstrained problem approximates the constraint problem. An extensive treatment of this topic can be found in [4]. In the sequel the successive unconstrained quadratic penalty minimization will be presented and the convergence properties will be expounded. The quadratic penalty function method consists of solving a sequence of problems in order to obtain a saddle point of the Lagrangian. Consider the equality constrained problem,

$$\min f(\mathbf{y}), \quad \mathbf{h}(\mathbf{y}) = \mathbf{0}, \qquad \mathbf{y} \in \mathcal{C}, \tag{10}$$

where $f : \mathcal{R}^n \rightarrow \mathcal{R}, \mathbf{h} : \mathcal{R}^n \longrightarrow \mathcal{R}^m$ are given functions, and $\mathcal{C}$ is a given subset of $\mathcal{R}^n$. The Lagrangian function of this problem is stated as,

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{y}) + \boldsymbol{\lambda}^T\, \mathbf{h}(\mathbf{y}). \tag{11}$$

A solution $\mathbf{y}^*$ of this optimization problem exists in the convex-concave saddle region of the Lagrangian function if the solution set is nonempty. At the center of the analysis is the augmented Lagrangian function $\mathcal{L}_a : \mathcal{R}^n \times \mathcal{R}^m \rightarrow \mathcal{R}$ given by,

$$\mathcal{L}_a(\mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{y}) + \boldsymbol{\lambda}^T\, \mathbf{h}(\mathbf{y}) + \frac{c}{2} \|\mathbf{h}(\mathbf{y})\|^2, \tag{12}$$

where c is a positive penalty parameter that induces a convexification of the saddle region. Following sequence of problems are being solved:

$$\min_{\mathbf{y}} \mathcal{L}_{a^k}(\mathbf{y}, \boldsymbol{\lambda}^k), \quad \mathbf{y} \in \mathcal{C} \tag{13}$$

where $\{\boldsymbol{\lambda}^k\}$ is a sequence in $\mathcal{R}^m$ and $\{c^k\}$ is a positive penalty parameter sequence that increases to ∞.

A. *Convergence*

The following proposition is the basic convergence result:

Proposition A: Assume that f and $\mathbf{h}$ are continuous functions, that $\mathcal{C}$ is a closed set, and that the constraint set $\{\mathbf{y} \in \mathcal{C} |\, \mathbf{h}(\mathbf{y}) = 0\}$ is nonempty. For $k = 0, 1, \ldots,$ let $\mathbf{y}^k$ be a global minimum of the problem:

$$\min \mathcal{L}_{a^k}(\mathbf{y}, \boldsymbol{\lambda}^k),\, \mathbf{y} \in \mathcal{C}, \tag{14}$$

where $\boldsymbol{\lambda}^k$ is bounded, $0 < c^k < c^{k+1}$ for all k, and $c^k \rightarrow \infty$. Then every limit point of the sequence $\mathbf{y}^k$ is a global minimum of the original problem (10).

Proof: Let $\bar{\mathbf{y}}$ be a limit point of the sequence $\{\mathbf{y}^k\}$. One has by the definition of $\mathbf{y}^k$

$$\mathcal{L}_{a^k}(\mathbf{y}^k, \boldsymbol{\lambda}^k) \leq \mathcal{L}_{a^k}(\mathbf{y}, \boldsymbol{\lambda}^k), \qquad \forall \mathbf{y} \in \mathcal{C} \tag{15}$$

If f^* signifies the optimal value, then following statement is true:

$$\begin{aligned} f^* &= \inf_{\mathbf{h}(\mathbf{y}=0),\mathbf{y}\in\mathcal{C}} f(x) \\ &= \inf_{\mathbf{h}(\mathbf{y})=0,\mathbf{y}\in\mathcal{C}} \left\{ f(\mathbf{y}) + \boldsymbol{\lambda}^{k^T}\mathbf{h}(\mathbf{y}) + \frac{c^k}{2}\|\mathbf{h}(\mathbf{y})\|^2 \right\} \\ &= \inf_{\mathbf{h}(\mathbf{y})=0,\mathbf{y}\in\mathcal{C}} \mathcal{L}_{a^k}(\mathbf{y}, \boldsymbol{\lambda}^k). \end{aligned} \tag{16}$$

Hence, by taking the infimum of the right-hand side of equation (16) over $\mathbf{y} \in \mathcal{C}$, $\mathbf{h}(\mathbf{y}) = \mathbf{0}$, following result is obtained

$$\mathcal{L}_{a^k}(\mathbf{y}^k, \boldsymbol{\lambda}^k) = f(\mathbf{y}^k) + \boldsymbol{\lambda}^{k^T}\mathbf{h}(\mathbf{y}^k) + \frac{c^k}{2}\|\mathbf{h}(\mathbf{y}^k)\|^2 \leq f^*. \tag{17}$$

The sequence $\{\boldsymbol{\lambda}^k\}$ is bounded which implies that it has a limit point $\bar{\boldsymbol{\lambda}}$. By taking the limit superior in the relation above and by using the continuity of f and $\mathbf{h}$, one obtains:

$$f(\bar{\mathbf{y}}) + \bar{\boldsymbol{\lambda}}^T\mathbf{h}(\bar{\mathbf{y}}) + \limsup_{k\to\infty}\frac{c^k}{2}\|\mathbf{h}(\mathbf{y}^k)\| \leq f^* \tag{18}$$

For $c^k \to \infty$, $\mathbf{h}(\mathbf{y}^k) \to 0$ is valid because of $\|\mathbf{h}(\mathbf{y}^k)\|^2 \geq \mathbf{0}$. So at the limit point $\mathbf{h}(\bar{\mathbf{y}}) = 0$ is valid. Otherwise the left-hand side would equal ∞.

B. Inequality Constraints

In order to treat inequality constraints in the framework of quadratic penalty approach, a conversion into equality constraints will be necessary. Consider the problem:

$$\min f(\mathbf{y}) \tag{19}$$

$$\text{subject to} \qquad h_1(\mathbf{y}) = 0, \ldots, h_m(\mathbf{y}) = 0, \tag{20}$$

$$g_1(\mathbf{y}) \leq 0, \ldots, g_r(\mathbf{y}) \leq 0. \tag{21}$$

The Lagrangian to this optimization problem is given by:

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{y}) + \boldsymbol{\lambda}^T\mathbf{h}(\mathbf{y}) + \boldsymbol{\mu}^T\mathbf{g}(\mathbf{y}), \tag{22}$$

$$\boldsymbol{\mu}^T\mathbf{g}(\mathbf{y}) = \mathbf{0}, \quad \boldsymbol{\mu} \geq \mathbf{0}, \quad \mathbf{g}(\mathbf{y}) \leq \mathbf{0}. \tag{23}$$

The vectorial complementarity

$$\boldsymbol{\mu}^T\mathbf{g}(\mathbf{y}) = \mathbf{0}, \quad \boldsymbol{\mu} \geq \mathbf{0}, \quad \mathbf{g}(\mathbf{y}) \leq \mathbf{0}, \tag{24}$$

can equivalently be expressed as:

$$\mu_j = \frac{1}{c}\mathrm{prox}_{\bar{C}}(\mu_j + c\,g_j(\mathbf{y})), \qquad j = 1, \ldots, r \tag{25}$$

or

$$g_j(\mathbf{y}) = \frac{1}{c}\mathrm{prox}_{\bar{C}}(\,g_j(\mathbf{y}) - c\,\mu_j), \qquad j = 1, \ldots, r \tag{26}$$

Here $y = prox_C(x)$ denotes the nearest point $y \in C$ to x. The corresponding augmented Lagrangian function that is being successively minimised can be obtained as:

$$\mathcal{L}_a^k = f(\mathbf{y}^k) + \boldsymbol{\lambda}^{k^T}\mathbf{h}^k + \frac{c^k}{2}\mathbf{h}^{k^T}\mathbf{h}^k + \frac{1}{2\,c^k}\sum_{j=1}^{r}\{(\mu_j^{k+1})^2 - (\mu_j^k)^2\}, \tag{27}$$

after several algebraical manipulations. And the update formulas are given by:

$$\mu_j^{k+1} = \mathrm{prox}_C(\mu_j^k + c^k\,g_j^k(\mathbf{y})), \qquad j = 1, \ldots, r \tag{28}$$

$$\lambda_j^{k+1} = \lambda_j^k + c^k\,h_j^k(\mathbf{y}), \qquad j = 1, \ldots, m \tag{29}$$

$m_1 = 1\,kg$	$m_2 = 1\,kg$	$\Theta_1 = 0.1\,kg\,m^2$	$\Theta_2 = 0.1\,kg\,m^2$
$a_1 = 0.4\,m$	$a_2 = 0.4\,m$	$g = 9.81\,\frac{m}{s^2}$	$\mu = 0.2$

TABLE I

MECHANICAL PARAMETERS OF THE TELESCOPE ARM.

The convergence proof in subsection (III-A) along with the extension of the augmented Lagrangian method to inequalities in subsection (III-B) provides a certificate of optimality if the requirements of proposition A are fulfilled.

C. Description of the Algorithm

The algorithm consists of three iterations which are embedded in each other. The most outer iteration is controlling the level nonlinearity of the system. The continuation parameter β is increased so that when β becomes one the system of equalities and inequalities fully represents the discretized system. The intermediate iteration performs for each given β a number of successive minimizations of the resulting augmented Lagrangian function $\mathcal{L}_{a^k}$ and updates after every iteration the dual multiplier vectors $\boldsymbol{\lambda}^k$, $\boldsymbol{\mu}^k$. The inner iteration performs the minimization of the augmented Lagrangian $\mathcal{L}_{a^k}$ for a given β and Lagrange multiplier vector by a modified conjugate gradients method. In investigating the optimal behaviour of hybrid mechanical system the gradient of the augmented Lagrangian with respect to $\mathbf{y}$ has been determined analytically. The continuation parameter β has been increased gradually to one and the intermediate systems of equations are also been partially minimized in order to approximate the optimal state and Lagrange multiplier trajectories iteratively. The penalty parameter c is increased as the value of β is gradually increased to one. In the initial stages of the optimization, where β and c are increased gradually, only the trajectories of the primal values are inherited to the next stage. As β approaches one, also the trajectories of the dual multipliers are inherited along with the trajectories of the primal values to the next stage.

A modified conjugate-gradients algorithm has been used in order to perform the minimizations of the augmented Lagrangian $\mathcal{L}_{a^k}$. the The conjugate gradient algorithm does not use prespecified conjugate directions, but instead computes the directions as the algorithm progresses.

IV. NUMERICAL RESULTS OF THE TELESCOPE ARM AND THE BIPED

One of the non-smooth mechanical system investigated in this section is depicted in figure 1. The mechanical parameters are given in table (I) and table (II) includes the given initial and desired end position which should be reached time-optimally.

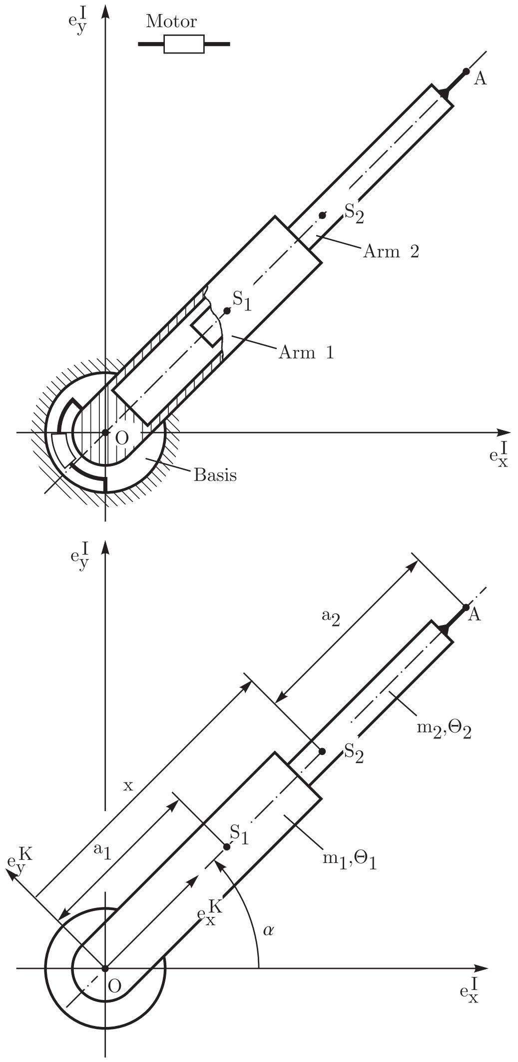

Fig. 1. telescope mechanism with one blockable degrees of freedom.

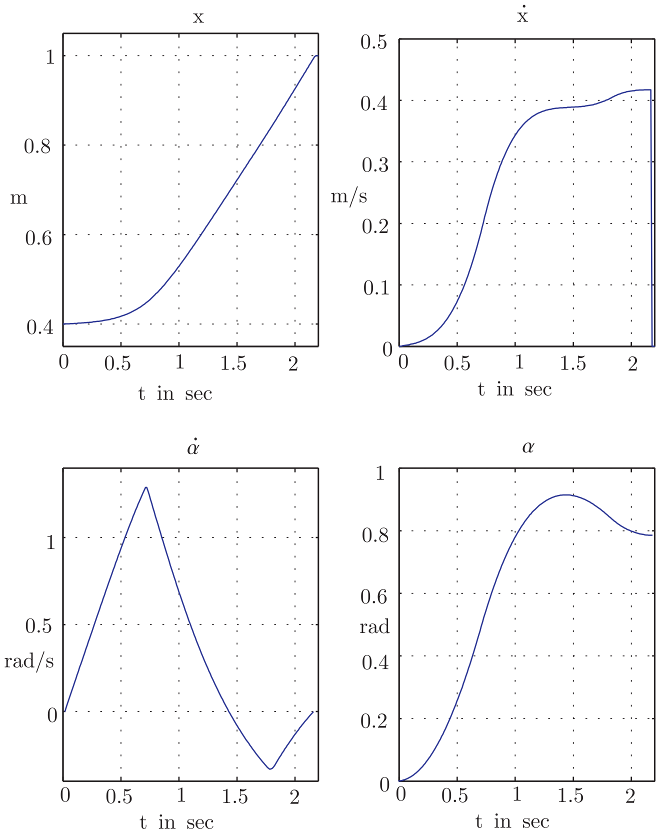

Fig. 2. (a) The linear position x, (b) linear velocity $\dot{x}$, (c) angular velocity $\dot{\alpha}$, (d) angular position α of the telescopearm.

$\alpha_0 = 0\,rad$	$\dot{\alpha}_0 = 0\,\frac{rad}{s}$	$x_0 = 0.4\,m$	$\dot{x}_0 = 0\,\frac{m}{s}$
$\alpha_f = \frac{\pi}{4}\,rad$	$\dot{\alpha}_f = 0\,\frac{rad}{s}$	$x_f = 1\,m$	$\dot{x}_f = 0\,\frac{m}{s}$

TABLE II

INITIAL AND FINAL CONDITIONS OF THE TELESCOPE ARM TRAJECTORY OPTIMIZATION.

In order to obtain the NLP, Moreau's time-stepping discretization scheme is used. The measure-differential inclusion that describes the dynamics is given by:

$$\begin{pmatrix} m_2 & 0 \\ 0 & \Theta \end{pmatrix} \begin{pmatrix} d\dot{x} \\ d\dot{\alpha} \end{pmatrix} - \begin{pmatrix} m_2\, x\, \dot{\alpha}^2 \\ -2 m_2\, x\, \dot{x} \dot{\alpha} \end{pmatrix} dt = \begin{pmatrix} d\Lambda_s \\ M\, dt \end{pmatrix}, \tag{30}$$

$$d\Lambda_s \in -\mu\, d\Lambda_u\, \mathrm{Sgn}(\dot{x}), \tag{31}$$

where $\Theta = \Theta_1 + \Theta_2 + m_1\, a_1^2 + m_2\, x^2$ and μ denote the friction coefficient.

This planar mechanical system has a fully actuated motor at the rotary link and the telescopic arm can only be blocked by means of a frictional break. This system can be considered as a hybrid system with explicit phase transitions. It is worth to note that by commanding the normal contact force at the blockable linear arm, the transitions between the 2-DOF mode and 1-DOF mode are controlled. The normal force measure is of impulsive character, so the system has controls of mixed measure and ordinary type. In order to formulate the equality and inequality conditions of the optimization problem the frictional contact will be decomposed into two complementarities making use of the following relations:

$$\begin{aligned} d\Lambda_R &= d\Lambda_s + \mu\, d\Lambda_u, \\ d\Lambda_L &= -d\Lambda_s + \mu\, d\Lambda_u, \\ \xi_R - \xi_L &= \dot{x}^+, \\ \xi_R \geq 0 & \quad \xi_L \geq 0, \\ d\Lambda_R \geq 0 & \quad d\Lambda_L \geq 0, \\ \xi_L\, d\Lambda_L &= 0, \\ \xi_R\, d\Lambda_R &= 0. \end{aligned}$$

Here $d\Lambda_u$ and $d\Lambda_s$ denote the normal and tangential contact differential measure forces, respectively. This decomposition

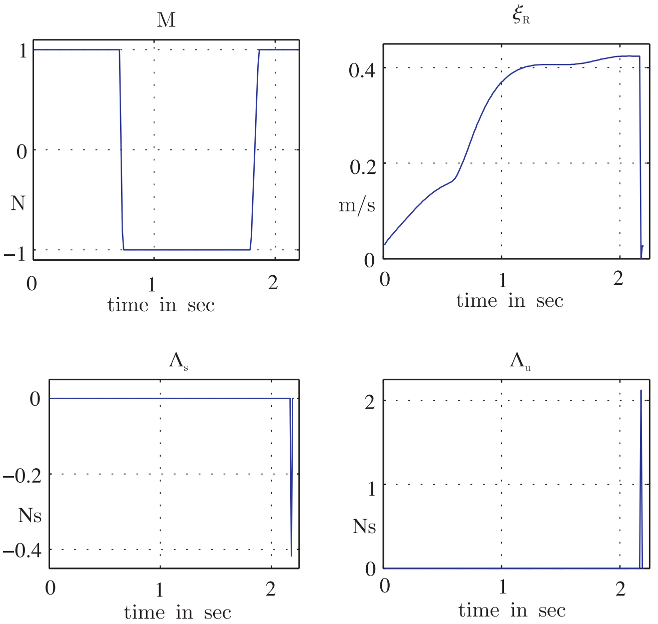

Fig. 3. The moment M, the right component ξ_r of the decomposed linear velocity, the tangential force measure Λ_s, the normal force measure Λ_u.

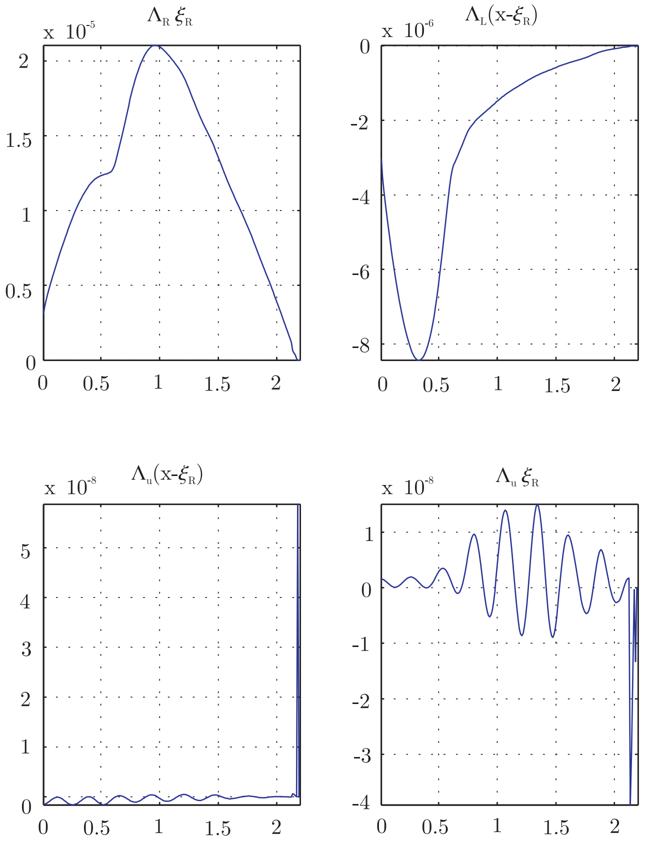

Fig. 4. The value of the complementarities during the course of the maneuver.

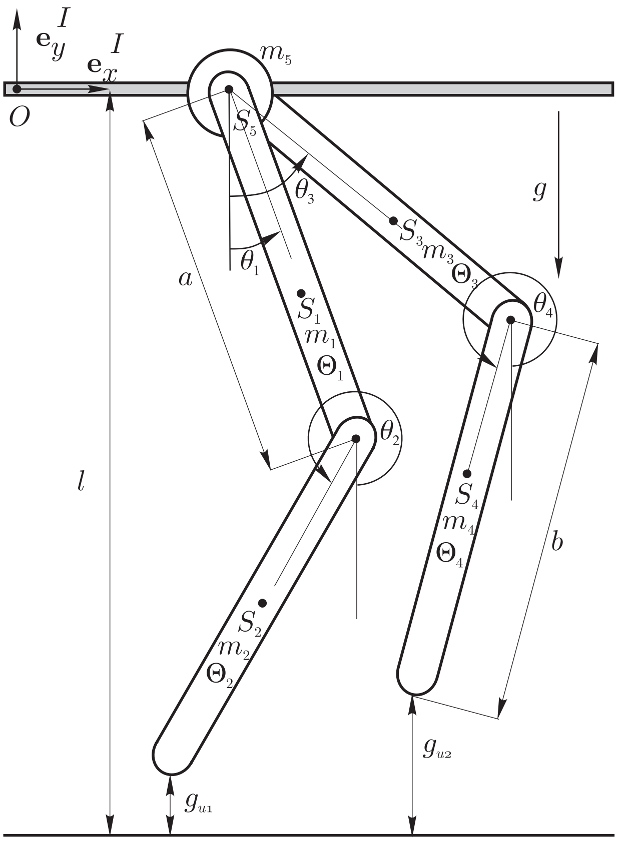

Fig. 5. The 5-DOF biped.

of the signum relation is due to Glocker [1]. In order to impose full blocking when the normal force is active, a blocking condition will be imposed such that when normal blocking force is present the relative contact velocity in the next moment reduces to zero:

$$\Lambda_u\,\xi_R = 0, \tag{32}$$

$$\Lambda_u\,(\xi_R - \dot{x}^+) = 0. \tag{33}$$

The four nonlinear complementarities, each consisting of one equality and 2 inequality constraints will be replaced by the Fischer-Burmeister function in the following manner:

$$x \perp y \iff fb(x,y) = \sqrt{x^2+y^2} - x - y = 0, \tag{34}$$

which has first been introduced in [6] in the framework of nonlinear programming. In addition there are lower and upper bounds to the moment that moves the arm and the normal impulses that can be applied. An equidistant discretization consisting of 200 points is used, and this resulted in an optimization problem in 1401 variables and 3210 inequality constraints. Figure (2) depicts the optimal state trajectories whereas figure (3) includes measure contact forces and the actuating moment M itself. The bang-bang like course of M indicates a time-optimal trajectory. The impactive breaking of the linear arm is clearly visible in the trajectories of $\dot{x}$ as well

$m_1 = 1\,kg$	$m_2 = 1\,kg$	$m_3 = 1\,kg$
$m_4 = 1\,kg$	$m_5 = 1\,kg$	$\Theta_1 = 0.05\,kg\,m^2$
$\Theta_2 = 0.05\,kg\,m^2$	$\Theta_3 = 0.05\,kg\,m^2$	$\Theta_4 = 0.05\,kg\,m^2$
$\Theta_5 = 0.05\,kg\,m^2$	$a = 0.4\,m$	$b = 0.4\,m$
$c = 0.6\,m$	$l = 0.6\,m$	$\epsilon_{u2} = 0.5$
$\epsilon_{u1} = 0.5$	$\epsilon_{s1} = 0$	$\epsilon_{s2} = 0$
$\mu_1 = 0.5$	$\mu_2 = 0.5$	$f_x = -5\,N$

TABLE III
MECHANICAL PARAMETERS OF THE 5-DOF BIPED.

as $d\Lambda_s$ and $d\Lambda_u$. At the end of the time-optimal maneuver, the blocking action induces an impactive transition from the 2-DOF mode to the 1-DOF mode. The second example includes a planar 5-DOF prototypical biped as a mechanical system. The biped consists of five rigid links one of which connects the two legs. The model is sketched in figure (5). The biped has nine different modes of operation. The modes of operation emanate from the fact that the leg contacts can be open or closed, or can be sliding or sticking when a particular contact is closed. Since there are two leg contacts, this results in nine modes. Transitions between all modes are possible, so the number of possible transitions are 81. It is obvious that the combinatorial complexity of the problem can not be tackled by mixed-integer type of programming. Table (III) summarizes the mechanical parameters, and table (IV) the initial and final states, respectively. The hip of the mechanism is restrained to move linearly along the x-axis. At each rotary joint controlled motors are assumed which are limited in the range -10 to 10 Nm. A number of 100 discretization points are used which resulted in an optimization problem with 1501 variables and 5422 inequality constraints. The mechanism is required to move time-optimally from the initial position where it rests with closed contacts, to the final position at $x = 1\,m$ and to come to rest in the same posture as in the initial situation. During the maneuver a constant force $f_x = -5N$ is applied that pulls the mechanism backwards. The response of the mechanism is to push itself forward using maximal possible torques in the joints and to brake itself at the desired end state. During the flight phase the legs are raised minimally from the floor. The optimization suggests a transition from the 1-DOF mode in which both legs stick to the flight mode where the system has 5-DOF. At the end there is a transition from the 5-DOF mode to the 1-DOF mode where it brakes the system. It is worth to mention that the left and right legs perform symmetrically in doing this jump. The trajectories can be seen in figure (6). Figure (7) depicts the contact forces at the legs during the course of the maneuver. Figure (8) presents the moments at the rotary joints where the initial maximal effort can be seen.

V. DISCUSSION AND CONCLUSION

A numerical method is presented for the determination of optimal trajectories for mechanical structure-variant systems. The method benefits from a sound modeling approach for

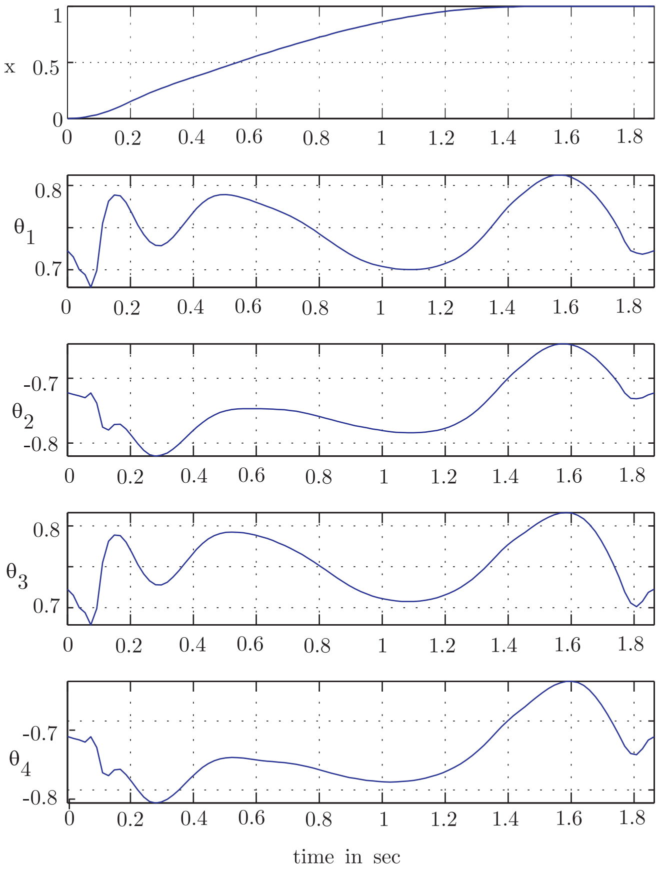

Fig. 6. Linear and angular trajectories.

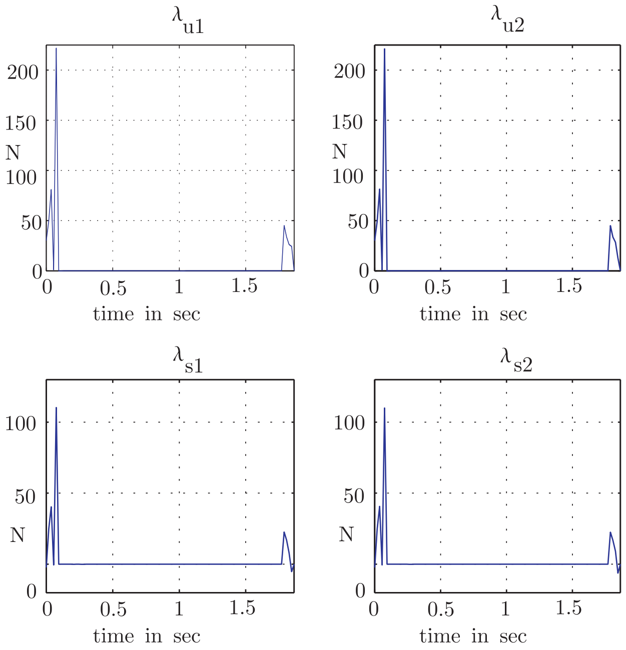

Fig. 7. Contact forces at the two legs.

$x_0 = 0\,m$	$x_f = 1\,m$	$\dot{x}_0 = 0\,\frac{m}{s}$	$\dot{x}_f = 0\,\frac{m}{s}$
$\theta_{10} = 0.73\,rad$	$\theta_{1f} = 0.73\,rad$	$\dot{\theta}_{10} = 0\,\frac{rad}{s}$	$\dot{\theta}_{1f} = 0\,\frac{rad}{s}$
$\theta_{20} = -0.73\,rad$	$\theta_{2f} = -0.73\,rad$	$\dot{\theta}_{20} = 0\,\frac{rad}{s}$	$\dot{\theta}_{2f} = 0\,\frac{rad}{s}$
$\theta_{30} = 0.73\,rad$	$\theta_{3f} = 0.73\,rad$	$\dot{\theta}_{30} = 0\,\frac{rad}{s}$	$\dot{\theta}_{3f} = 0\,\frac{rad}{s}$
$\theta_{40} = -0.73\,rad$	$\theta_{4f} = -0.73\,rad$	$\dot{\theta}_{40} = 0\,\frac{rad}{s}$	$\dot{\theta}_{4f} = 0\,\frac{rad}{s}$

TABLE IV

INITIAL AND FINAL STATES OF THE 5-DOF BIPED.

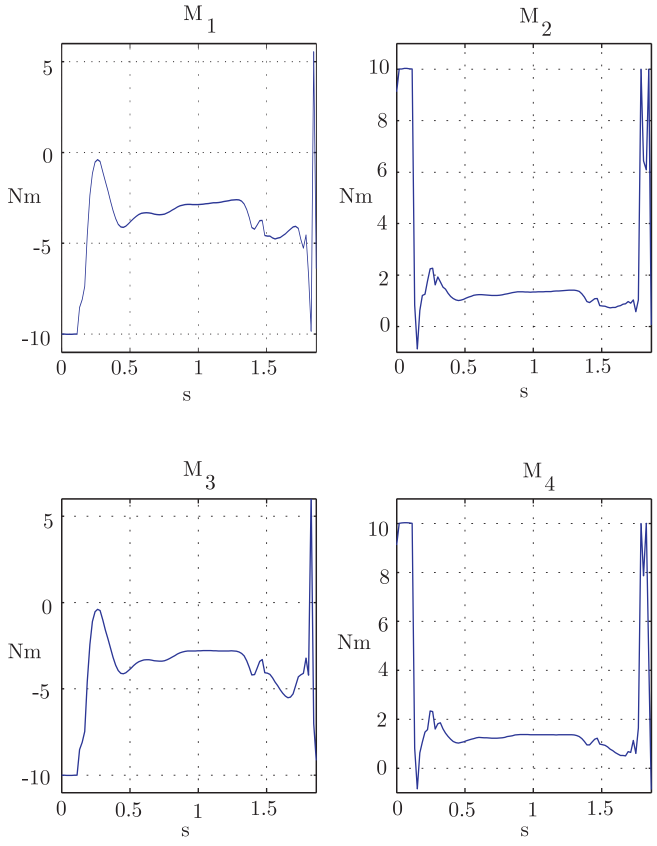

Fig. 8. The actuating moments of the biped.

structure-variant systems based on the measure-differential inclusion representation. The novel feature is that contrary to other NLP schemes for optimization problems with different phases, characterized by different system dynamics, mixed-integer programming is not necessary and the parameters of event or phase transitions need not to be optimized separately in the optimization. As a consequence, the location and time of phase transitions where the system changes DOF is not pre-specified but is determined as an outcome of the optimization. The main features of the introduced SUMT-based method can be summarized as follows:

1. Though the underlying system might undergo structure-variant phase changes such as impactive phase transitions and stick-slip transitions a mixed-integer programming approach is not necessary.
2. The MDI representation encompasses the unilateral and frictional contacts of the mechanical system.
3. Pre-estimation of the adjoint variables is not necessary which are closely related to the Lagrange multipliers.
4. The sufficiency condition for a convergence to a at-least locally optimal solution is the existence of a nonempty feasible set deriving from the convergence proof.
5. The method minimizes over modes as well and chooses a sequence of modes and transitions which possess a certificate of optimality.

The global convergence of the SUMT for solving the underlying problem is enhanced by the convexification induced utilizing augmented Lagrangian based SUMT.

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