

Optimality Conditions for Mechanical Impacts

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In this report the necessary conditions for optimal non-smooth control of rigid body mechanical systems with multiple impacts will be stated in accordance with Pontryagin's Minimum Principle. Criteria based on the Weierstrass-Erdmann (WE) conditions and contemporary impact theory will be established in order to assess the optimality of an impact. The determination of post-impact state and costate values from the pre-impact values will be possible in some cases.

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Mechanical systems subject to inequality (or unilateral) constraints and impact effects have recently been the object of significant interest in the mechanical engineering and applied mathematics scientific communities, in parallel the analysis and control of hybrid dynamical systems is an active area of investigation in the systems and control community. From the point of view of calculus of variations on which optimal control builds up on, this requires to consider the variations of piecewise-smooth maximizing trajectories. Mechanically, these non-smooth trajectories are obtained with impacts. If the allowable set of variations is extended to include piecewise-smooth trajectories, it is evident that a reduction of the cost functional can further be achieved in comparison to the case where only smooth optimal trajectories are considered, which constitute a subset of the piecewise-smooth maximizing trajectories. Indeed, there are even cases where a solution among the smooth extremizing trajectories does not exist because of reachability space considerations. In order to discuss impacts for a general rigid body system with Newton impact law and Coulomb friction, a linear complementarity problem (LCP) can be formulated in terms of impulses and contact kinematics variables that will enable the determination of the post-impact state. The solution of the LCP can be evaluated together with the Weierstrass-Erdmann condition so that the optimality of the impact can be assessed. The equation of motion of a general scleronomic mechanical system is given by:

$$M(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau}) - \mathbf{W}_s(\mathbf{q})\boldsymbol{\lambda}_s - \mathbf{W}_u(\mathbf{q})\boldsymbol{\lambda}_u = 0 \quad (1)$$

Here \mathbf{q} , $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ represent the position, velocity and acceleration in the generalized coordinates, respectively; $\boldsymbol{\tau}$ denotes the control inputs to the system; $M(\mathbf{q})$ is the $n \times n$ symmetric positive-definite mass matrix; \mathbf{h} is a $n \times 1$ vector that includes all velocity and position dependent forces and external forces of finite magnitude; $\mathbf{W}_s(\mathbf{q})$ is a $n \times m$ matrix of generalized friction force directions; $\mathbf{W}_u(\mathbf{q})$ is a $n \times m$ matrix of generalized normal force directions, $\boldsymbol{\lambda}_s$ is the tangential contact force and $\boldsymbol{\lambda}_u$ is the normal contact force. The impact equations of a rigid body system with impacts and friction can be obtained by integrating equation (1) over the impact time and is given by:

$$\begin{aligned} \int_{\{t_{imp}\}} M(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau}) - \mathbf{W}_s(\mathbf{q})\boldsymbol{\lambda}_s - \mathbf{W}_u(\mathbf{q})\boldsymbol{\lambda}_u dt &= M(\mathbf{q})(\dot{\mathbf{q}}^+ - \dot{\mathbf{q}}^-) - \mathbf{W}_s(\mathbf{q})\Lambda_s - \mathbf{W}_u(\mathbf{q})\Lambda_u \\ &= M(\mathbf{q})(\dot{\mathbf{q}}^+ - \dot{\mathbf{q}}^-) - \mathbf{W}(\mathbf{q})\Lambda = 0 \end{aligned} \quad (2)$$

A superscripted $^+$ denotes the post-impact value of the related entity, and analogously superscripted $^-$ denotes the pre-impact value of the related entity. Here Λ_s is the tangential contact impact impulse and Λ_u is the normal contact impact impulse. The tangential and normal local kinematics need to be defined in order to relate the contact distance to the set-valued force element. If it is known a priori that at the closing of the contacts an impact with stick will occur, the desired relation between the pre-impact and post-impact velocities can be obtained by utilising the Newton impact law, contact kinematic relations and equation(2) as:

$$\dot{\mathbf{q}}^+ = (\mathbf{E} - M^{-1}(\mathbf{q})\mathbf{W}(\mathbf{q})(\mathbf{W}^T(\mathbf{q})M^{-1}(\mathbf{q})\mathbf{W}(\mathbf{q}))^{-1}(\boldsymbol{\epsilon} + \mathbf{E})\mathbf{W}^T(\mathbf{q}))\dot{\mathbf{q}}^- = \mathbf{K}(\mathbf{q})\dot{\mathbf{q}}^- \quad (3)$$

where \mathbf{E} and $\boldsymbol{\epsilon}$ denote the identity matrix and a diagonal matrix with restitution coefficients as entries, respectively. The matrix \mathbf{K} will be of fundamental importance in further analysis in analyzing the Weierstrass-Erdmann corner conditions because it relates the variations of the post-impact generalized velocities to variation of the pre-impact velocities. In the case where the impact state is assumed a priori, \mathbf{K} is a linear transformation. In general the solution(s) of the LCP constitute the set of possible such transformations \mathbf{K} . In the following a piecewise-smooth extremal trajectory will be considered, so that $\dot{\mathbf{q}}$ will be continuous except at one time in the interval (t_o, t_f) . We assume that $\dot{\mathbf{q}}$ has a discontinuity at some point $t_{imp} \in (t_o, t_f)$, which is not fixed, nor it is usually known in advance where the discontinuity occurs. The problem is to find a necessary

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condition that must be satisfied by the extrema of the functional at the point of discontinuity on velocity level. The functional to be minimized is:

$$\mathcal{J}_a(\mathbf{q}, \dot{\mathbf{q}}, \tau, t) = \int_{t_o}^{t_{imp}^-} g(\mathbf{q}, \dot{\mathbf{q}}, \tau) dt + \int_{t_{imp}^+}^{t_f} g(\mathbf{q}, \dot{\mathbf{q}}, \tau) dt, \quad (4)$$

subject to:

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, \tau) = -\mathbf{M}^{-1}(\mathbf{q}) \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, \tau) \quad (5)$$

$$\mathbf{q}(t_o) = \mathbf{q}_o, \quad \mathbf{q}(t_f) = \mathbf{q}_f, \quad \dot{\mathbf{q}}(t_o) = \dot{\mathbf{q}}_o, \quad \dot{\mathbf{q}}(t_f) = \dot{\mathbf{q}}_f, \quad (6)$$

$$\mathbf{g}_u^- = 0, \quad \mathbf{g}_u^+ = 0, \quad \gamma_u^- = 0, \quad \gamma_u^+ = 0. \quad (7)$$

Here $\mathbf{g}_u(\mathbf{q}), \gamma_u(\mathbf{q}_{imp}, \dot{\mathbf{q}}_{imp}), \dot{\gamma}_u(\mathbf{q}_{imp}, \dot{\mathbf{q}}_{imp}, \ddot{\mathbf{q}}_{imp})$ denote normal contact distance, velocity and acceleration, respectively. The normal contact distance has to be zero just before and after the impact. It is an end point constraint to the first part and an initial point constraint to the second part. The normal contact velocity and acceleration should be equal to zero for those contacts which remain closed after the impact. It is assumed that g has continuous first and second order partial derivatives with respect to all of its arguments, and that $t_o, t_f, \mathbf{q}(t_o)$ and $\mathbf{q}(t_f)$ are specified without loss of generality. The introduction of a first-order differential equation system will enable the formulation of the Hamiltonian functional:

$$\mathcal{H} = g + \lambda^T \mathbf{f} + \mathbf{p}^T \mathbf{u}. \quad (8)$$

where \mathbf{u} represents the generalized velocities, so that the costate equations can then be expressed in following terms:

$$\dot{\lambda}^T = -\frac{\partial \mathcal{H}}{\partial \dot{\mathbf{q}}} = -\frac{\partial g}{\partial \dot{\mathbf{q}}} - \lambda^T \mathbf{M}^{-1}(\mathbf{q}) \frac{\partial \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}, \tau)}{\partial \dot{\mathbf{q}}} - \mathbf{p}^T, \quad \dot{\mathbf{p}}^T = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \quad (9)$$

Performing the variations following conditions are obtained for a set of impacts to be optimal if also a subset of the contacts remain closed after impact time:

- Variations due to impact time:

$$\mathcal{H}^- - \mathcal{H}^+ = (\mu_g^- - \mu_g^+ \epsilon) \mathbf{W}_u^T * \dot{\mathbf{q}}^- + \underline{(\mu_\gamma^+ \nabla_{\dot{\mathbf{q}}} \gamma_u^+ + \mu_\gamma^+ \nabla_{\dot{\mathbf{q}}} \dot{\gamma}_u^+) \mathbf{K} * \dot{\mathbf{q}}^-} + \underline{(\mu_\gamma^+ \nabla_{\mathbf{q}} \gamma_u^+ + \mu_\gamma^+ \nabla_{\mathbf{q}} \dot{\gamma}_u^+) * \dot{\mathbf{q}}^+} \quad (10)$$

- Variations due to impact location:

$$* \mathbf{p}^{T-} - * \mathbf{p}^{T+} = (\mu_g^- + \mu_g^+) \mathbf{W}_u^T + \underline{\mu_\gamma^+ \nabla_{\mathbf{q}} \gamma_u^+ + \mu_\gamma^+ \nabla_{\mathbf{q}} \dot{\gamma}_u^+}$$

- Variation of pre-impact velocity:

$$* \lambda^{T-} \mathbf{K}^{-1} = * \lambda^{T+} + \underline{(\mu_\gamma^+ \nabla_{\dot{\mathbf{q}}} \gamma_u^+ + \mu_\gamma^+ \nabla_{\dot{\mathbf{q}}} \dot{\gamma}_u^+)} \quad (11)$$

- Variation of post-impact acceleration:

$$\underline{\mathbf{W}_u^T \delta \ddot{\mathbf{q}}^+} = 0 \quad (12)$$

where $\mu_\gamma^+, \mu_\gamma^-, \mu_g^+, \mu_g^-$ represent Lagrange multipliers with values greater or equal zero, because the constraints $\mathbf{g}_u^-, \mathbf{g}_u^+, \gamma_u^+, \dot{\gamma}_u^+$ can be formulated such as to have values less or equal to zero at impact times. The underlined parts of the necessary conditions vanish when all contacts open immediately after the impact moment.

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